

A decision model for an inventory system with two compound Poisson demands

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ABSTRACT

A real inventory system for single item with specific demand characteristics motivates this works. The demand can be seen as two types of independent demand, where compound Poisson process describes the characteristics of each demand. The first type of demand is rarely occurred with relatively large size, while the second type of demand is often happened with relatively small size. In order to maintain inventory level, every time the first type of demand occurs a replenishment of stock is conducted which follows order-up-to-level inventory policy. In order to find the optimal inventory decision for that system, a mathematical model of the system is developed with the objective to minimize expected total inventory cost. Some of model assumptions are infinite replenishment, deterministic lead time, and completely backlogged shortages. To solve the model, it is then divided into two sub-problems and classical optimization technique is employed to help find the solution of each sub problem.

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1. Introduction

In each supply chain or supply network, every entity at a certain echelon must be connected to more than one entity at its lower echelon. For example, a factory must be connected to several distribution centers, or a distributor center must be connected to several retailers. In the supply chain or supply network, a common mechanism between two entities at different echelon is that a lower-echelon entity sends a request to higher-echelon entity to deliver goods and this request is fulfilled from the inventory kept in the higher-echelon entity, i.e. a distribution center sends a request to factory to deliver goods and this request is fulfilled from the factory inventory. Due to this mechanism, inventory management can be considered as an essential part in the supply chain management field and numerous researches in the past have been focusing on the supply chain optimization including inventory decision (Matinrad et al., 2013). In this era of big data, all industrial systems have the opportunity to obtain various information which are useful in making decisions. Finished product inventory system is one of the systems that have been widely studied by researchers (Zipkin, 2000; Axsäter, 2015). In this inventory research, information about demand is one of the important parameters to be obtained through various emerging analytic techniques (Tiwari et al., 2018; Wang et al., 2016).

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Recently, we conducted big data analytics on the transaction data of a final product in a factory warehouse. After going through various analytics processes, we cannot conclude that the demand for this product is following certain probability distribution. However, the demand for this product consists of two types of independent demands, each of which follows a different probability distribution. If these two types of demands are compared based on the occurrence frequency, the first type of demand has much less frequent occurrence than the second one does. Meanwhile, when they are compared based on the demand size, the first type of demand is relatively much larger than the second one. This demand pattern motivated us to develop various mathematical models that can be used to make inventory decisions in these situations. Some research works on inventory model in the past assumed that the occurrence of demand follows Poisson process. For simultaneously modeling the demand occurrence and its size, a compound Poisson process is utilized. The earliest work that used compound Poisson process as demand model was the work of Feldman (1978), where it is applied to a continuous review (s, S) inventory system. Following this research, some other models were proposed, such as time dependent continuous review model with finite planning horizon (Banerjee et al., 1985), continuous review model for one warehouse and many retailers (Axsäter et al., 1994), model with stochastic lead time (Song, 1994) and base-stock system for multi item inventory (Song, 1998). Similar approaches of using the compound Poisson process for modeling demand have been applied in the area of production and inventory problem (de Kok, 1985; Altiok & Ranjan, 1995; Liu & Cao, 1999; Song, 2000). Recently, Haji et al. (2018) discussed a decision related price and inventory, where the demand was modeled as Poisson demand, inside a VMI (vendor managed inventory) system. Johansson et al. (2020) dealt with control of inventory and shipment in distributor where the demand was also modeled as compound Poisson Demand. It is noted that all the previous mentioned inventory research works involving both single and multi-item have always considered the demand following a single compound Poisson process. To the best of our knowledge, there is no research involved more than one compound Poisson process to model demand before Boxma et al. (2014) and Boxma et al. (2016). In their works, the demand was alternated between high demand and low demand periods. In different demand periods, the demands are modeled using different compound Poisson processes. Therefore, this demand pattern is totally different with the demand pattern considered in this paper. A mathematical model to handle a compounded demand pattern is being proposed and solved in this paper. The remainder of this paper is structured into following sections. Section 2 contains the problem formulation. Section 3 explains the mathematical model and its solutions. Section 4 presents some numerical cases and sensitivity analysis. Finally, Section 5 will conclude this research work.

2. Problem Formulation

2.1. Demand Patterns

As mentioned in the introduction, this single item inventory system handles two types of demand. In this paper, let define demand X and demand Y be the first and the second types of demand, respectively. The occurrence and size of demands can be illustrated in Fig. 1. It is noted that the subscript i represents the order of occurrence, for example X_2 is order size of the second occurrence and Y_3 is order size of the third occurrence of Y .

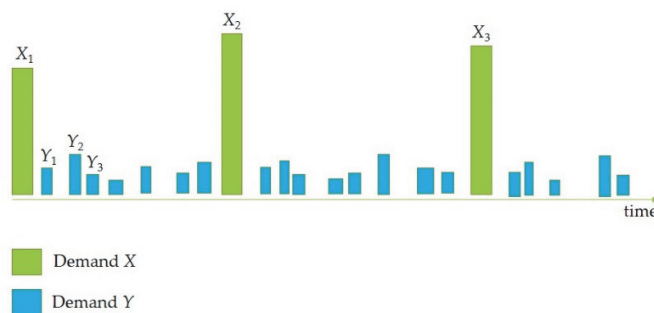


Fig. 1. Illustration of occurrence and size of demands

Demand X occurs following a Poisson process with parameter λ_X and the magnitude follows a general distribution G_X with the pdf function $f_X(\cdot)$. Demand Y occurs following a Poisson process with parameter λ_Y and the magnitude follows a general distribution G_Y with the pdf function $f_Y(\cdot)$. Based on the description in the introduction, demand X occurs compared with demand Y , i.e. $\lambda_X \leq \lambda_Y$. Also, the expectation of order size of demand X is higher than expectation of order size of demand Y , i.e. $E(X) \geq E(Y)$. Following definitions in the previous paragraph, each type of demand is a compound Poisson process. Therefore, demand X is a stochastic process $\{Z_1(t), t \geq 0\}$, in which

$$Z_1(t) = \sum_{i=1}^{N_1(t)} X_i, t \geq 0, \quad (1)$$

where $\{N_1(t), t \geq 0\}$ is a Poisson process with parameter λ_X , and $\{X_i, i \geq 1\}$ are independently and identically distributed random variables with cumulative function G_X that is independent of $\{N_1(t), t \geq 0\}$. It is well known from Ross (2014) that the expectation and variance of $Z_1(t)$ are

$$E[Z_1(t)] = \lambda_X t \cdot E[X], t \geq 0, \quad (2)$$

$$\text{Var}[Z_1(t)] = \lambda_X t \cdot E[X^2], t \geq 0. \quad (3)$$

Similarly, demand Y is also a stochastic process $\{Z_2(t), t \geq 0\}$, in which

$$Z_2(t) = \sum_{i=1}^{N_2(t)} Y_i, t \geq 0, \quad (4)$$

where $\{N_2(t), t \geq 0\}$ is a Poisson process with parameter λ_Y , and $\{Y_i, i \geq 1\}$ are independently and identically distributed random variables with cumulative function G_Y that is independent of $\{N_2(t), t \geq 0\}$. Also, it can be derived that

$$E[Z_2(t)] = \lambda_Y t \cdot E[Y], t \geq 0, \quad (5)$$

$$\text{Var}[Z_2(t)] = \lambda_Y t \cdot E[Y^2], t \geq 0. \quad (6)$$

Adding X and Y , the expectation and variance of total demand during time t are then formulated as

$$E[D(t)] = t(\lambda_X E[X] + \lambda_Y E[Y]), \quad (7)$$

$$\text{Var}[D(t)] = t(\lambda_X E[X^2] + \lambda_Y E[Y^2]). \quad (8)$$

2.2. Inventory Policy

The inventory policy used for dealing with this situation is similar to the periodic review inventory policy, where inventory replenishment is carried out, periodically. However, instead of placing the order at every fixed period of time as in the standard periodic review inventory policy, the order will be placed whenever demand X occurs. Hence, the time between two consecutive replenishments (T) is no longer deterministic, but is probabilistic. From the demand pattern definition, it is known that demand X occurs following Poisson process with parameter λ_X . Hence, the length of an inventory cycle T will follow an exponential distribution with parameter λ_X . Therefore, the expectation and variance of T are expressed as

$$E[T] = 1/\lambda_X, \quad (9)$$

$$\text{Var}[T] = 1/\lambda_X^2. \quad (10)$$

Decision variable of this policy is order up-to-level (I), where the level of inventory will be raised up to I units every time it is replenished.

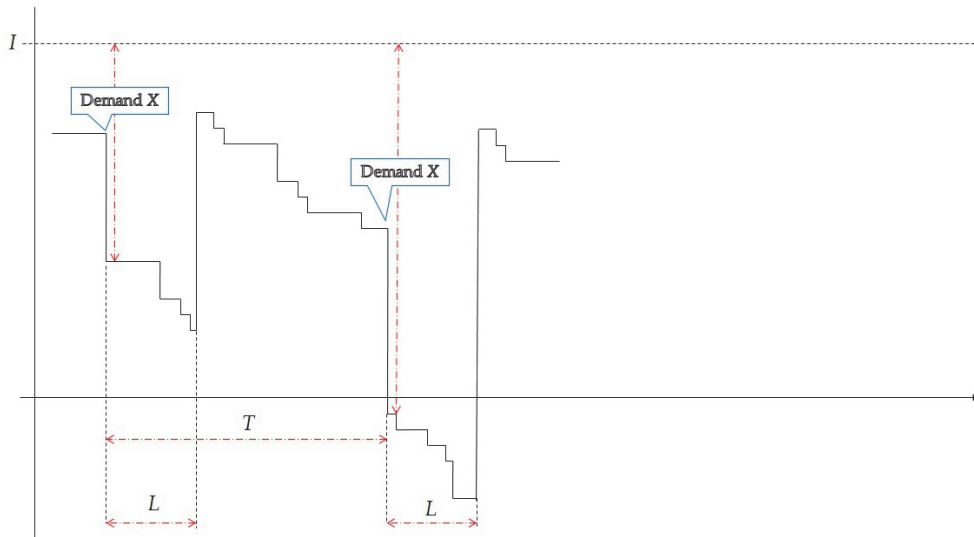


Fig. 2. Illustration of inventory policy and inventory level progression

2.3. Optimization Problem and Assumptions

The optimization problem considered here is to decide the value of I for minimizing the expected total inventory cost for a single item inventory system that has demand pattern defined in section 2.1 and uses inventory policy as defined in section 2.2. Cost components that are included in the total cost function are ordering cost (with unit cost c_o), holding cost (with unit cost c_h), and backorder cost (with unit cost c_s). Some assumptions that are relevant to be considered are as follows:

- infinite replenishment rate,
- non zero lead time (L) is non zero and deterministic, and
- shortages are allowed and are completely backlogged.

3. Model Development

With the purpose to solve the optimization problem, we propose to decompose this problem into two sub problems. Each sub problem is associated only to a single type of demand: the sub problem X is associated to demand X and the sub problem Y is associated to demand Y . Sub problem X has a single decision variable named I_X and sub problem Y also has a single decision variable named I_Y . The relationship between these decision variables and decision variable of the original problem is

$$I = I_X + I_Y. \quad (11)$$

It is noted that the objective function for both sub problem X and sub problem Y are also the expected total inventory costs, i.e. TC_X and TC_Y , respectively, with their corresponding cost components. The relationship among these objectives with the objective of original problem is as follows

$$TC(I) = TC_X(I_X) + TC_Y(I_Y). \quad (12)$$

Relationship between the sub- problems and the original problem is illustrated in Fig. 3.

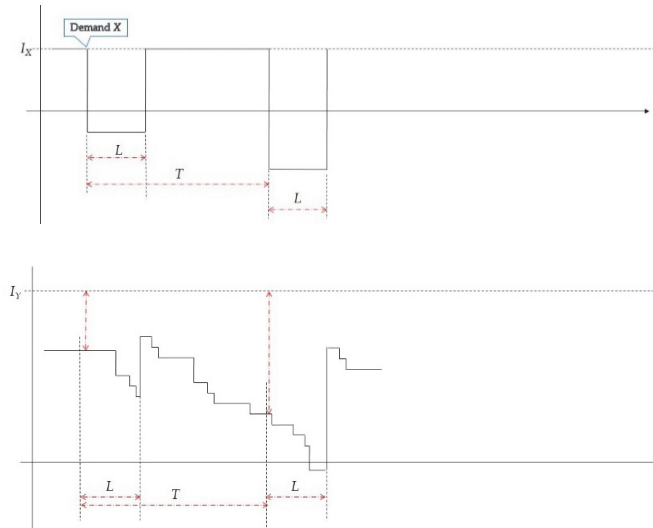


Fig. 3. Inventory level in the sub problem X (above) and sub problem Y (below)

3.1. Optimization of Sub Problem X

In this sub problem, T is a random variable with expectation value expressed in Eq. (9). We define that the cost components of this sub problem are only backorder and holding costs. Backorder is possible to happen only during the lead time L . It happens whenever the demand X is greater than I_X . Therefore, the expected backorder cost during one cycle of replenishment can be formulated as

$$c_s \int_{I_X}^{\infty} L(X - I_X) f_X(X) dX. \quad (13)$$

During lead time, inventory is possible to occur if the demand X is smaller I_X . However, after the order arrives the inventory will remain at I_X until the next demand X occurs. Therefore, the expression of the expected holding cost during one replenishment cycle is

$$c_h \left[I_X (E[T] - L) + \int_0^{I_X} L(I_X - X) f_X(X) dX \right]. \quad (14)$$

The objective function of this sub problem can be obtained by combining Eq. (13) and Eq. (14), and dividing the expression with the expected value of T . Therefore, the objective of sub problem X is to minimize the following expectation expression

$$TC_X(I_X) = \lambda_X c_h \left[I_X \left(\frac{1}{\lambda_X} - L \right) + \int_0^{I_X} L(I_X - X) f_X(X) dX \right] + \lambda_X c_s \int_{I_X}^{\infty} L(X - I_X) f_X(X) dX. \quad (15)$$

Since this problem is a single variable optimization problem, after analyzing the optimization condition as shown in Appendix A, it is obtained that the optimum decision for sub problem X can be obtained through following equation:

$$\int_0^{I_X^*} f_X(X) dX = 1 - \frac{c_h}{(c_h + c_s)} \cdot \frac{1}{L}. \quad (16)$$

3.2. Optimization of Sub Problem Y

In order to analyze this sub problem, let focus on a single replenishment cycle with a period of length T . Instead of starting the cycle at the occurrence of demand X , the cycle can be started at the arrival time of a replenishment order. Similar to sub problem X , T is also a random variable with expected value defined in Eq. (9).

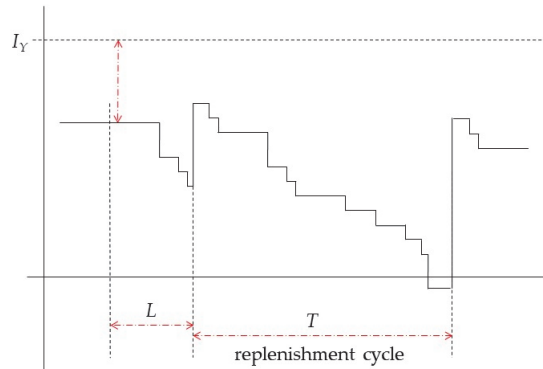


Fig. 4. Replenishment cycle in sub problem Y

Based on Fig. 4, it can be seen that the level of inventory at the cycle beginning is equal to I_Y minus total demand Y during lead time L . So, the expectation of inventory level at the beginning of a cycle can be expressed as

$$E[I_0] = I_Y - \lambda_Y L \int_0^{\infty} Y f_Y(Y) dY. \quad (17)$$

Let T_0 be the length of the time period from the start of a replenishment cycle until when the inventory level reaches zero, $T_0 \leq T$. Mathematically, it can be derived that

$$E[I_0] = \lambda_Y \cdot E[T_0] \int_0^{\infty} Y f_Y(Y) dY. \quad (18)$$

The following expression can be obtained after associating Eq. (17) and Eq. (18).

$$E[T_0] = \frac{I_Y}{\lambda_Y \int_0^{\infty} Y f_Y(Y) dY} - L. \quad (19)$$

Expectation of inventory holding cost in one replenishment cycle can be approximately calculated as the area of a right triangle with $E[T_0]$ as its base and $E[I_0]$ as its height. Therefore, the expected holding cost of one replenishment cycle can be determined by the following equation

$$C_h = \frac{c_h}{2} \left(\frac{I_Y^2}{\lambda_Y \int_0^{\infty} Y f_Y(Y) dY} - 2I_Y L + \lambda_Y L^2 \int_0^{\infty} Y f_Y(Y) dY \right). \quad (20)$$

Expectation of backorder amount in one replenishment cycle can be approximately calculated as the area of a right triangle with $E[T] - E[T_0]$ as its base and $E[S]$ as its height in which the expected backorder amount at the end of a replenishment cycle, i.e. $E[S]$, can be calculated as follows,

$$E[S] = \lambda_Y \left(\frac{1}{\lambda_X} + L \right) \int_0^{\infty} Y f_Y(Y) dY - I_Y. \quad (21)$$

Hence, the expectation of backorder cost in one replenishment cycle can be approximately expressed as follows,

$$C_s = \frac{c_s}{2} \left(\lambda_Y \left(\frac{1}{\lambda_X} + L \right)^2 \int_0^{\infty} Y f_Y(Y) dY - 2 \left(\frac{1}{\lambda_X} + L \right) I_Y + \frac{I_Y^2}{\lambda_Y \int_0^{\infty} Y f_Y(Y) dY} \right). \quad (22)$$

The objective of this sub problem is similar to the objective of the sub problem X , which is the expected total cost per unit time. It can be obtained by combining ordering, holding, and backorder cost in one

cycle, and dividing the resulting expression by the expected value T . After some algebra, it can be found that the expression of the objective is presented as Eq. (23) as follows,

$$TC_Y(I_Y) = \lambda_X c_o + \lambda_X \frac{c_s}{2} \left[\lambda_Y \left(\frac{1}{\lambda_X} + L \right)^2 \int_0^\infty Y f_Y(Y) dY - 2 \left(\frac{1}{\lambda_X} + L \right) I_Y + \frac{I_Y^2}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} \right] + \lambda_X \frac{c_h}{2} \left[\frac{I_Y^2}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} - 2 I_Y L + \lambda_Y L^2 \int_0^\infty Y f_Y(Y) dY \right]. \tag{23}$$

This problem is also a single variable optimization problem. Therefore, after analyzing the optimization condition as shown in Appendix B, it is obtained that the optimum decision for sub problem Y can be derived as presented below:

$$I_Y^* = \left[\frac{c_s}{(c_s + c_h)} \frac{1}{\lambda_X} + L \right] \lambda_Y \int_0^\infty Y f_Y(Y) dY. \tag{24}$$

4. A case example and sensitivity analysis

A case example is presented here in order to demonstrate the model applicability. Let consider a situation of two compound Poisson process in an inventory system, in which the demand X occurs with $\lambda_X = 1/60$ days and the demand Y occurs with $\lambda_Y = 1/30$ days. The magnitude of demand X follows uniform distribution $U[100, 200]$ and the magnitude of demand Y follows uniform distribution $U[10,20]$. The unit cost component of ordering, holding, and backorder are $c_o = \$ 50000/\text{order}$, $c_h = \$ 1/\text{unit}/\text{day}$, and $c_s = \$ 15/\text{unit}/\text{day}$, respectively. Using Eq. (16) and Eq. (24), the optimum decision of each sub problem can be determined as follow.

$$\int_{100}^{I_X^*} f_X(X) dX = 1 - \frac{1}{(1+15)} \cdot \frac{60}{5} \Leftrightarrow I_X^* = 125. \tag{25}$$

$$I_Y^* = \left[\frac{1}{(1+15)} \frac{1}{1/60} + 5 \right] \frac{1}{30} \int_{10}^{20} Y \frac{1}{20} dY = 30.63. \tag{26}$$

Therefore, the value of I is 155.63 units. Using these value of decision variables, the expected total cost for each sub problem are \$ 133.59/day and \$ 847.40/day, respectively. Hence, the total cost is \$980.99/day. Sensitivity analyses with some changes in model parameter are studied in the following section. At first, the effect of λ_X and λ_Y on the optimal decision variables and corresponding total cost will be examined. Table 1 shows the effect of those parameters. It shows that whenever λ_X increases, I_X^* , I_Y^* , I^* the total costs increase (TC_X , TC_Y , and TC). Meanwhile, changes in λ_Y has no effect on I_X^* and TC_X .

Table 1
Effect of λ_X and λ_Y on decision variables and objective function

λ_X	λ_Y	I_X^*	I_Y^*	I^*	TC_X	TC_Y	TC
1/80	1/30	0.00	40.00	40.00	14.06	643.75	657.81
1/70	1/30	112.50	35.31	147.81	120.59	730.69	851.28
1/60	1/30	125.00	30.63	155.63	133.59	847.40	980.99
1/50	1/30	137.50	25.94	163.44	146.95	1011.72	1158.67
1/40	1/30	150.00	21.25	171.25	160.94	1259.38	1420.31
1/60	1/40	125.00	22.97	147.97	133.59	843.88	977.47
1/60	1/35	125.00	26.25	151.25	133.59	845.39	978.98
1/60	1/30	125.00	30.63	155.63	133.59	847.40	980.99
1/60	1/25	125.00	36.75	161.75	133.59	850.21	983.80
1/60	1/20	125.00	45.94	170.94	133.59	854.43	988.02

Moreover, whenever λ_Y increases, I_Y^* , I^* , TC_Y and TC increase. Next, the effects of c_s and c_h on the optimal decision variables and corresponding total cost will be investigated. The results are presented in Table 2. It shows that when c_h increases, both I_X^* and I_Y^* decrease, so that I^* also decreases. However, all total cost expressions (TC_X , TC_Y , and TC) increase. It also shows that when c_s increases, I_X^* , I_Y^* , I^* , and all total cost expressions increase.

Table 2

Effect of c_h and c_s on decision variables and objective function

c_h	c_s	I_X^*	I_Y^*	I^*	TC_X	TC_Y	TC
0.50	15	161.29	31.53	192.82	93.46	840.59	934.05
0.75	15	142.86	31.07	173.93	117.54	844.05	961.59
1.00	15	125.00	30.63	155.63	133.59	847.40	980.99
1.25	15	107.69	30.19	137.88	142.18	850.64	992.82
1.00	13	114.29	30.36	144.64	121.10	847.26	968.36
1.00	14	120.00	30.50	150.50	127.67	847.33	975.00
1.00	15	125.00	30.63	155.63	133.59	847.40	980.99
1.00	16	129.41	30.74	160.15	138.99	847.45	986.44
1.00	17	133.33	30.83	164.17	143.94	847.50	991.44

Finally, the effect of expected demand size of X and Y on the optimal decision variables and corresponding total cost will be examined. The results are presented in Table 3, in which the same uniform distribution is assumed for both distributions and only the expectation of X and Y are presented in the table.

Table 3

Effect of $E[X]$ and $E[Y]$ on decision variables and objective function

$E[X]$	$E[Y]$	I_X^*	I_Y^*	I	TC_X	TC_Y	TC
150	15	125.00	30.63	155.63	133.59	847.40	980.99
150	30	125.00	61.25	186.25	152.34	861.46	1013.80
150	60	125.00	122.50	247.50	189.84	889.58	1079.43
150	90	125.00	183.75	308.75	227.34	917.71	1145.05
150	150	125.00	306.25	431.25	302.34	973.96	1276.30

5. Conclusions

An inventory decision model for a single item inventory system with two compound Poisson demands has been developed in this paper. The model helps to decide the order-up-to-level for minimizing the expected total inventory cost. By decomposing the model into two independent sub problems and using classical optimization, the optimal solution of the model can be obtained. For future works, it is suggested to investigate other inventory policies to tackle the same situation. Other possible extensions are to consider partial backorder and to generalize the model to consider more than two types of demand.

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Appendix A

In order to obtain the optimal decision for the sub problem X , the first derivative of Eq. (15) will be derived as follows,

$$\frac{d}{dI_X} TC_X(I_X) = \lambda_X c_h \left(\frac{1}{\lambda_X} - L \right) + \lambda_X c_h L \int_0^{I_X} f_X(X) dX - \lambda_X c_s L \int_{I_X}^{\infty} f_X(X) dX. \quad (A1)$$

Setting the first derivative equal to zero as the necessary condition of minimization problem and dividing all expressions with λ_X , yields,

$$c_h \left(\frac{1}{\lambda_X} - L \right) + c_h L \int_0^{I_X} f_X(X) dX - c_s L \int_{I_X}^{\infty} f_X(X) dX = 0. \quad (A2)$$

Substituting the integral part in the last term with the general definition of probability density function, Eq. (A2) can be rewritten as

$$c_h \left(\frac{1}{\lambda_X} - L \right) + c_h L \int_0^{I_X} f_X(X) dX - c_s L \left(1 - \int_0^{I_X} f_X(X) dX \right) = 0, \quad (A3)$$

$$L(c_h + c_s) \int_0^{I_X} f_X(X) dX = L(c_h + c_s) - c_h \frac{1}{\lambda_X}, \quad (\text{A4})$$

$$\int_0^{I_X} f_X(X) dX = 1 - \frac{c_h}{(c_h + c_s)} \cdot \frac{1/\lambda_X}{L}. \quad (\text{A5})$$

The optimal value for I_X can then be obtained from Eq. (A5). It is noted that this optimal value is the minimum value, since the second derivative of the expected total cost expression for this sub problem is greater than zero.

$$\frac{d^2}{dI_X^2} TC_X(I_X) = \lambda_X L f_X(I_X) (c_h + c_s) > 0. \quad (\text{A6})$$

Appendix B

In order to obtain optimal decision for the sub problem Y the first derivative of Eq. (23) is determined as follows,

$$\frac{d}{dI_Y} TC_Y(I_Y) = \lambda_X \frac{c_s}{2} \left[-2 \left(\frac{1}{\lambda_X} + L \right) + \frac{2I_Y}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} \right] + \lambda_X \frac{c_h}{2} \left[\frac{2I_Y}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} - 2L \right]. \quad (\text{B1})$$

Setting the first derivative equal to zero as the necessary condition of minimization problem and dividing all expressions with λ_X , yields the following,

$$\frac{c_s}{2} \left[-2 \left(\frac{1}{\lambda_X} + L \right) + \frac{2I_Y}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} \right] + \frac{c_h}{2} \left[\frac{2I_Y}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} - 2L \right] = 0. \quad (\text{B2})$$

Eq. (B2) can be modified as

$$c_s \left[- \left(\frac{1}{\lambda_X} + L \right) + \frac{I_Y}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} \right] + c_h \left[\frac{I_Y}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} - L \right] = 0. \quad (\text{B3})$$

$$\frac{(c_s + c_h) I_Y}{\lambda_Y \int_0^\infty Y f_Y(Y) dY} = \frac{c_s}{\lambda_X} + (c_s + c_h) L. \quad (\text{B4})$$

Hence, the optimal value for I_Y can be obtained from

$$I_Y^* = \left[\frac{c_s}{(c_s + c_h) \lambda_X} + L \right] \lambda_Y \int_0^\infty Y f_Y(Y) dY. \quad (\text{B5})$$

The solution in Eq. (B5) is minimal, since the sufficient condition of this optimization problem can be confirm as presented below.

$$\frac{d^2}{dI_Y^2} TC_Y(I_Y) = \frac{\lambda_X}{\lambda_Y} \frac{(c_s + c_h)}{\int_0^\infty Y f_Y(Y) dY} > 0. \quad (\text{B6})$$

