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Many to many hub and spoke location routing problem based on the gravity rule

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C H R O N I C L E

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This paper examines the spoke and hub location decisions in a routing problem. To minimize the total cost, the study analyzes on how to locate the spokes, hubs and the allocation of spoke nodes to hub nodes, the routing among the nodes and the number of vehicles assigned to each hub thoroughly. As there might be no facility assigned to some points, unsatisfied demands must be distributed to other nodes with available facilities. Furthermore, the realized demand is determined by considering the perceived utility of each path, using The Gravity rule. For this purpose, the proposed nonlinear model is transformed into a linear programming model, where some tightening rules and preprocessing procedures are applied, and also the sequential and integrated approaches are developed to solve the problem. In the sequential method, spokes are allocated, and hubs are selected based on the location of the spokes, after which the routing in the local tour is determined. Meanwhile, in the integrated approach, the aggregated model is solved. A heuristic is presented to address the integrated model. Numerical experiments are run on both approaches, to compare both, and obtain insights from the model.

1. Introduction

The transportation of parcels or public postal services is one of the most important applications of freight transportation. Overall, in this application, instead of the direct connection between the origin and the destination points, the hub and spoke networks are used together. Shipping is either full-truckload or less-than-truckload in the freight industry. Full-truckload is usually used for large-size shipment while less-than-truckload handles smaller packages. According to the literature, the share of less-than-truckload has been moving up recently. In the United States and China, the value of the share is about 32$ and 30$, respectively. The annual growth rate in China was about 10% between 2010 and 2012 (Bruns et al., 2000; Ni et al., 2016). Due to its importance, freight transportation providers have always tried to provide better services along minimizing their costs. More information about the freight transportation is provided by Crainic (1999).

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A hub and spoke model has been presented in this paper. First, some facilities are chosen (the number is unknown). Second, some of them are selected as hubs and also the allocation of non-hub facilities to hub facilities are determined. Afterwards, the routing among the nodes for transferring the freights and the number of vehicles assigned to each hub are defined, in order to minimize transportation costs, lost demand penalty and facility set-up costs. It is better to state that in this paper, the non-hub facilities are defined as spokes or domestic facilities. A transportation network including 13 nodes (shown in circles) is depicted in Fig. 1. Triangle-circle nodes are locations in which spokes are built to serve demands while square-triangle-circle nodes are considered as hubs.

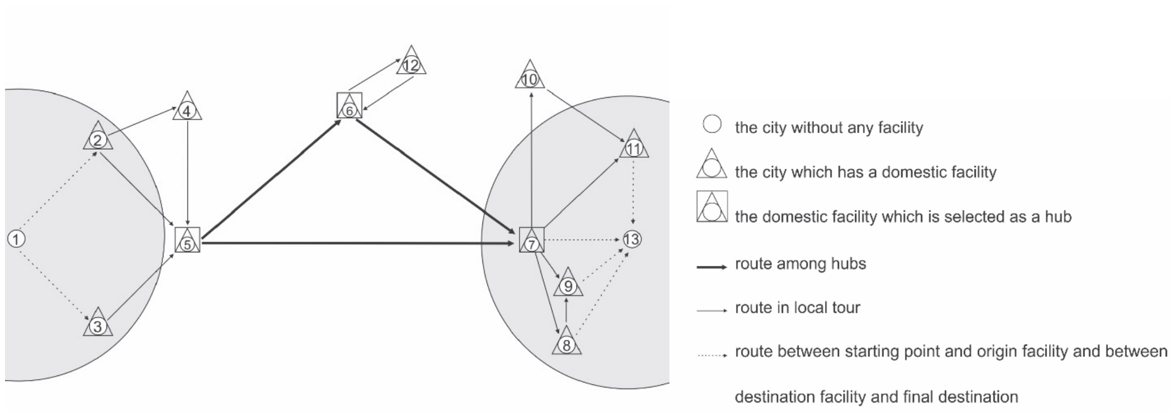

Fig. 1. The graphical representation of nodes in a transportation system

Assuming that domestic facilities have "sphere of influence", the demand is served by the facilities in their sphere of influence. It means that if the spoke (domestic) facility is farther away from the node with no facility, the demand is not transferred. Also, if there is no facility in the sphere of influence of one node, the demand is considered as a lost demand. For instance, in Fig. 1, the facilities located in the nodes 2 and 3 are in the sphere of influence of node 1. So the demand of node 1 can be distributed between these two nodes. On the other hand, the demand of node 1 is not willing to be distributed among other nodes, since the distance between those nodes and node 1 is relatively significant.

As shown in Fig. 1, the hub and spoke network is not considered as a fully connected network. This means that based on the solution of the optimization model, the links between spoke and hub should be selected. The hubs are fully connected. It is assumed that each freight will travel its route through one or two hubs and no two non-hub nodes could be served by a direct link if these nodes are allocated to different hubs. This is to reduce the incurred cost, as the consolidated flow among the hub network benefits from the economy of scale. Apparently, if the origin and destination facilities are hub themselves, the journey is considered as "One Leg".

Consider the freight can be transferred from node 1 to 13. As there is no facility in node 1 as a starting point, this demand should be distributed among node 2 and node 3 which have available facilities. As the perceived utilities of node 2 and node 3 are not the same, so the gravity rule resulted in the probability can be applied. The journey of the freight from the facility on node 2 or node 3 can be continued from different routes. In the first stage, the leg of 2-5 or 3-5 is selected to move into hub number 5, and after that leg 5-7 or 5-6-7 can be elected into hub number 7. From the final destination view (node number 13), all the nodes 8,9,11 and 7 are in its sphere of influence, so that the freight can be delivered from each of these points. The route 1-2-5-7-9-13 indicates that the freights should be transferred from the starting point 1 to final destination 13. So the requester of this service chooses node 2 as an Origin Facility and node 9 as a destination facility. When the freight arrives at node 2, it continues the journey using hub 5 and 7. After that, it arrives at destination facility 9 and reaches its final destination which is node 13. If there is a demand from the starting point 2 to point 9 as the final destination in this network, the route 2-5-7-9 indicates that the starting point is well equipped with facilities in a way that one is interested in taking that route through hubs 5 and 7, getting the right facilities in its ultimate destination.

In order to minimize the burden costs on travels, we have focused on locating the facilities and hubs simultaneously in this paper. The requester of a freight transportation determines among the origin facilities and destination facilities. In other words, when a freight arrives at the origin facility, the system / or service provider decides which route should be elected. It is a real case problem in which the freight should be transported by the hub and spoke network, but the requester for this transportation decides the origin and destination facility. For example, when a hub and spoke network is used to transport a parcel, firstly, the requester should transfer the parcel to the node which has a facility. It then needs to decide which node has a preference, and also when it wants to take the parcel from the destination facility. Apparently, it would have some preferences, but between the origin and destination facilities, the service provider should choose the route with the minimum transportation cost.

The first stage of the problem is to locate the facilities as spokes, the same as P-Median problem. Minimization of the total costs including set-up and penalty costs is the objective of the problem. It should be mentioned that while the model incorporates the customers' perceived utility, the sole decision maker of the model is the transportation company, not the customer. As a result, the customers from nodes left without any facility may travel to the nearby nodes with facilities, or would be treated as a "lost demand" case. The second part is known as the many-to-many- hub location-routing problem. The objective of this stage is to find the hub locations grounded in locating the domestic facilities, as explained above. At this stage, the local journeys for serving the nodes which are not hubs should be generated and then they need to be connected to the hubs. This will determine the route of connecting many origin facilities to many destination facilities by considering the all the costs.

Several studies in the literature have focused on the formulation of discrete hub location problem. This concept exists in many-to-many dispersal networks such as Cargo, Post, Telecommunications and Airlines, which all the flows between nodes are directed through hub facilities. Searching the optimal location and fulfilling the demand points by the right route between the origin and destination, are the main concerns of the hub problem. Hubs are expected to be middle facilities to consolidate and sort the shipments to reduce the cost of transportation.

The second part of our problem is also related to location-routing problems which were initiated by Watson-Gandy and Dohrn (1973). The many-to-many-hub-location-routing-problem was first introduced by Nagy and Salhi (1998). They designed a two-level system including vehicle and terminal routes in a way that the parcels which the customer wishes to send are picked up and the parcels that he/she wishes to receive are supplied by a number of identical vehicles in and out of the terminal. Locational and routing decisions should be made at the first and second level of the system, respectively.

Burns et al. (2000) investigated the location problem of Swiss Postal Services. They did not explicitly consider the routing decision, but they embedded the cost of routing in a location problem. Wasner and Zäpfel (2004) developed a nonlinear model in order to provide a solution for Austrian Mail Delivery System as a real case. A parallel approach was applied for solving the model. Varied local search routines linked with feedback loops were used in this approach. Çetiner et al. (2010) developed a two-phase method for solving a unified hub location and routing problem in mail delivery service. Mail offices are multiply allocated to the identified hub locations in the first phase. Then, the distance between points will be changed by giving the routes in hub regions at the second phase. The method is repeated between phases by updating the distances applied in the problem to achieve a route-matched hub configuration. Karaoglan et al. (2012) considered simultaneously two constraints in which that the hub-to-hub routes in pickup and delivery were not allowed in their location-routing problem. A new model for the many-to-many-location-routing problem was developed by Camargo et al. (2013). They solved the problem by decomposing tailored benders decomposition. A branch-and-cut algorithm was proposed by Rodríguez-Martín et al. (2014) for deciding on the nodes' location, nodes' allocation and routing among the nodes which were allocated to the same hubs. The objective of the model was to minimize the total transportation cost.

It is deemed that the nearest facility should serve the service in a classic P-Median problem which is called Proximity Assumption. It sounds logical that the closest facility should serve the demand. In this case, the customer's choice and attraction of facilities are not considered. Moreover, this assumption lets customers use only one facility which is called "All or Nothing" behaviour. This assumption does not support the real world cases because of the following reasons:

For more details, please refer to (Drezner & Drezner, 2011; Zabihi et al., 2016).

- If the location of the facility slightly shifts, it may radically shift the demand to another facility.
- It is unclear that due to the incompleteness of information, customers will use the same facility in demand point even if the attractiveness is unique.
- There is a Type C error in the aggregation literature as the population is often spatially aggregated and approximately represented by the centre of the demand point (Francis et l., 2009; Francis et al., 2000; Hillsman & Rhoda, 1978).
- Due to that, facilities could vary in attractiveness, customers may like a facility which is not the nearest one.
- A facility may be chosen for other purposes by the customer (Carling & Håkansson, 2013).

A more actual model for explaining the customers' behaviour could be developed by applying The Gravity Rule (Reilly, 1931) or the later (Huff, 1964, 1966). According to The Gravity Rule, the probability that a customer chooses a facility depends on the attractiveness and decreasing function of distance to a facility. It is worth to note that well-known models were reanalyzed by gravity rule rather than proximity rule by recent researchers.

Drezner and Drezner (2001) proposed the model for airline hub selection problem in which travellers have different choices to reach the destination by passing through different hubs. The proportional of the customers who select the definite hub is calculated by The Gravity Rule. The optimal location was analysed by Serra and Colomé (2001) in a competitive atmosphere, by considering different aspects of the customers' behaviours like distance costs and transportation costs. A P-Median problem on a network and plane was analysed and solved by Drezner and Drezner (2006, 2007), by relaxing the assumption that every user should be served by the nearest facility. Instead, user's patronage was basically divided among all facilities by applying The Gravity Rule. Azofra et al. (2007) studied the placement of sea rescue resources based on gravitational models which allowed us to define individual and zonal distribution models. A model in which the market capture is maximized with the assumption of less travel and waiting time, which are the major concerns of the customer to choose a facility, was proposed for locating the facilities by Marianov et al. (2008). The gravity rule was also applied in a Traveling Salesman Problem in order to compute the profit by Erdoğan et al. (2010). A facility's Attractiveness and its distance to the customer are the basis for attracting the portion of the profit from the customer by each visited facility. A school network problem was considered by Haase and Müller (2013) to maximize the utility of all students by incorporating capacity and budget constraints. Aros-Vera et al. (2013) proposed the model to define the location of the park and ride facilities so that their usage is maximized. Commuters can choose among the car and ride facilities and their own cars to travel to their destination. Their behavior follows The Gravity Rule. A mathematical model was developed by Castillo-López and López-Ospina (2015), addressing the location and size of the new school with the purpose of optimizing academic performance based on the students' behavior. Ai et al. (2015) presented a model for the location, allocation and configuration problem of emergency resources in a maritime emergency system. In their model, the probabilistic distribution of emergency demand based on The Gravity Model is considered. A model for "Domestic Facility and Hub Location" in transportation network was proposed by Khosravi and Jokar (2017). The objective of the model is to minimize the transportation, lost demand penalty and facility set up costs considering that the number of hubs and facilities which are unknown. Moreover, there is a compatibility between the gravity rule and the distribution of demands among domestic facilities and hubs.

In all of the above-mentioned articles, the location of spoke nodes are deterministic and given, and the demand between origin and destination facility is also fixed. In this paper, we relaxed this assumption and changed the problem of hub location to the hub and spoke location. As the location of spokes is not specified in the problem, the demand between them is also not given. So the proposed model must decide about the location of the non-hub nodes and the distribution of demand between the nodes which are selected as spoke nodes that follows The Gravity Rule.

This paper is inspired by all of the above literature in the following areas:

- Developing an Integrated model to find the location of hubs, non-hub (spoke) nodes, routing and the number of vehicles assigned for each hub simultaneously which has never been covered/studied as far as the authors investigated.
- Using the gravity function to distribute potential demand among the domestic facilities to provide more realistic and applicable model.
- Applying linearization techniques to transform a non-linear model to a linear form and adding some constraints to tighten the model.
- Proposing a specific upper bound for evaluating the performance of the integrated model and approach to a solution.
- Considering the real distance between nodes in routing problems which is directly dependant on the route.
- Analysing the model and solving with an exact and heuristic method.

The remainder of the paper is organized as follows. In section 2, we have introduced the formulation and defined the model. In this section, the two approaches are applied for modeling the problem. The first one is a two-sub problem sequential model and the second one is the integrated. In section 3, the linearization technique and tightening formulation are proposed. In section 4, the heuristic is developed for solving the integrated model. In section 5, we have reported the computational results on Turkish network data test and in section 6, we have concluded and proposed topics for future research.

2. Definitions and problem formulation

This model is a mixed integer non-linear formulation which is based on the combination of two problems: one is a model proposed by Khosravi and Jokar (2017) for facility location based on The Gravity Model and the other is a mathematical model proposed by de Camargo et al. (2013) for the many-to-many-hub location-routing problem.

Let N be the set of locations which exchange flows; P the candidate to become spoke facility and K the candidate that can be selected as hubs. In general, $P \subseteq N$ and $K \subseteq P$, however we will assume that $N \equiv P$ and $K \equiv P$. Therefore, all of the demand nodes are potential candidates to be a spoke, and all of the spoke nodes can be chosen as a hub. For all pairs of nodes i and j, $\mathbf{W_{ij}}$ represents the flow demand from the "starting point" i to the "final destination" j which should be routed through two spoke facility or can be considered as a lost demand. In this problem, transportation costs are proportional to the distance which means the triangle inequality holds and normally $\mathbf{W_{ij}} \neq \mathbf{W_{ji}}$. We have:

- $i, j \in N$: denote origin and destination nodes
- D_{ij}: denotes the distance between nodes i and j
- $m, n, u, v \in P$: interchangeably denote spoke nodes
- $p, t, x, z, k \in K$: denote for the hubs

- u_{ijmn}: denotes the utility of origin facility of "m", destination facility of "n" for the flow from the starting point of "i "to the final destination of "j" node
- F_m^S: denotes fixed cost of establishing a facility as spoke at node $m \in$ N
- F_m^h: denotes the fixed cost to upgrade an existing facility to a hub at node $m \in$ P
- α: denotes the inter-hub discount factor

Due to economies of scale, the trip cost for moving between a hub and a spoke or between two spoke facilities is more than the cost between hubs.

Also, C is a unit transportation cost as per each kilometer in local tour between domestic facilities and hub. V is the set of available vehicles, λ_{uv} is the traveling time of arc uv, T is the maximum time allowed for the tours, cc_{uv} is the cost of constructing arc uv, c_l is the cost of assigning vehicle l, c_{mn}^{pt} is the cost of demand wa_{mn} *and* wa_{nm} for interhub connection between p and t $c_{mn}^{pt} = (wa_{mn}c_{pt} + wa_{nm}c_{tp}) \times \alpha$ Where c_{pt} is the unit transportation cost of inter-hub connection and wa_{mn} is the level of demand from m as an origin facility to n as a destination facility.

Apparently, as there may be other alternatives to fulfill the customers' needs, not all demands are served by network facilities. In this case "lost demand" is defined as any demand which is out of sphere of influence and not in any other. In order to minimize the lost demand we define a penalty in the model and denote with B. The formulation uses the following variables:

Integer variables $\mathrm{z_m^s} \in \{0,1\}$ to indicate if node m is either selected to set up spoke facility $m \in N$ ($\mathrm{z_m^s} = 1$) or not ($\mathrm{z_m^s} = 0$) and z_{mk}^h to indicate if node m is either allocated to hub k $\left(\mathrm{z}_{mk}^{\mathrm{h}} = 1\right)$ *or not* ($\mathrm{z_{mk}^h} = 0$) and $\mathrm{z_{kk}^h} = 1$ if node k is selected as a hub, and otherwise it equals zero. $q_{kl} \in \{0,1\}$ to indicate if vehicle l assigns to hub k ($q_{kl} = 1$) or not ($q_{kl} = 0$). $y_{uv}^{kl} \in \{0,1\}$ to indicate if vehicle l of hub k used arc uv in its tour ($y_{uv}^{kl} = 1$) or not ($y_{uv}^{kl} = 0$). $x_{mn}^{kt} \in \{0,1\}$ to indicate if the flow between m and n ($m < n$) is routed from k and t in this order. $xx_{uv}^{mnkl} \in \{0,1\}$ to indicate if the flow between m and n which served by vehicle l of hub k is used arc uv ($xx_{uv}^{mnkl} = 1$) or not ($xx_{uv}^{mnkl} = 0$). $p_{\mathrm{u}}^{kl} \in \{0,1\}$ to indicate if node u is served by vehicle l of hub k ($p_{\mathrm{u}}^{kl} = 1$) or not ($p_{\mathrm{u}}^{kl} = 0$). $f_{uv}^{klm} \in \{0,1\}$ to indicate if vehicle l of hub k used arc uv to serve node m ($f_{uv}^{klm} = 1$) or not ($f_{uv}^{klm} = 0$).

This problem involves the following two interdependent sub-problems:

- How many and where the spoke facilities are required?
- How many and where the hubs are required? How should the spokes be allocated to the hub? How many vehicles will be assigned to each hub? In which order the nodes should be visited on the local tours of each hub?

There are two approaches to model and solve the problem: two sequential sub-problem and integrated one. Formerly, at first, we decide about the location of the spoke facility and in the second sub-problem, based on the location of spoke facility the remaining of the problem is modeled and solved. In this approach, when the first sub-problem is solved, the only discussion is about the tradeoff between the setup cost of spoke facility and lost demand cost. So in this sub-problem, the transportation cost is not considered. In the next (or second) approach, the model can be solved in an integrated manner. Although the integrated model will be more complex, but the result avoids local optimum.

2.1. Sub-problem 01

In this sub-problem, from n number of candidate locations, some will be selected to set up spoke facilities. This number is not fixed in prior and will be defined by solving this sub-problem. The objective function is to minimize the costs which composed of the domestic facility set up cost and the penalty for lost demand. The transportation cost, from the starting point to origin facility and from destination facility to final destination, is not related to the service provider and hence not considered in this model. The distribution of flow among the nodes which are spoke facilities adhere to gravity rule. It means that the proportion of flow from *i* to *j* who selected m as the origin facility and *n* as the destination facility is a function of attractiveness and distance which is defined here as the utility. The utility is depicted as u_{ijmn} ; therefore the proportion of passengers traveling from *i* to *j* and selecting facility m as origin and n as destination facility is:

$$\frac{u_{ijmn}}{\sum_{u,v} u_{ijkl} y_u^s y_v^s}. \tag{1}$$

The level of demand from *m* as an origin facility to *n* as a destination facility is equal to:

$$wa_{mn} = \sum_{i,j} w_{ij} \frac{u_{ijmn}.y_m^s y_n^s}{\sum_{uv} u_{ijuv} y_u^s y_v^s}, \tag{2}$$

and the lost demand is:

$$\left(\sum_{i,j} W_{ij} \text{-} \sum_{m,n} \sum_{i,j} w_{ij} \frac{u_{ijmn} y_m^s y_n^s}{\sum_{u,v} u_{ijuv} y_u^s y_v^s}\right). \tag{3}$$

So the lost demand cost is equal to:

$$B \times \left(\sum_{i,j} W_{ij} \text{-} \sum_{m,n} \sum_{i,j} w_{ij} \frac{u_{ijmn} y_m^s y_n^s}{\sum_{u,v} u_{ijuv} y_u^s y_v^s}\right). \tag{4}$$

The objective function is:

$$Z_{01} = \sum_m F_m^S y_m^s + B * \left(\sum_{i,j} W_{ij} - \sum_{m,n} \sum_{i,j} w_{ij} \frac{u_{ijmn} y_m^s y_n^s}{\sum_{u,v} u_{ijuv} y_u^s y_v^s} \right), \tag{5}$$

and there is no constraint in the model.

2.2. Sub-problem 02

At this sub-problem, the hubs will be selected among the facilities which were extracted from the previous stage. The route from the origin facility to the destination facility is determined, and also the number of vehicles assigned to each hub is fixed. The objective function at this stage is minimizing the cost which includes the fixed cost (hub set-up cost and constructing arc *uv*), the allocation vehicle to hub and the transportation cost (in hub network and local tour). In this sub- problem, the level of demand from m as an origin facility to n as a destination facility is equal to Eq. (2). As clearly addressed in above formula, it’s highly dependent on the outcome of sub problem 01.

$$\min \sum_k F_k^h . z_{kk}^h + \sum_{l,u,v,k} cc_{uv} . y_{uv}^{kl} + \sum_{k,l} q_{kl} . c_l + \sum_{\substack{m,n,k,t \\ m<n \\ k \neq t}} c_{mn}^{kt} . x_{mn}^{kt} + \sum_{u,v,m,n,k,l} wa_{mn} . xx_{uv}^{mnkl} . d_{uv} . c \tag{6}$$

$$\sum_k z_{mk}^h = 1 \tag{7}$$

$$z_{mk}^h \leq z_{kk}^h \quad m \neq k \tag{8}$$

$$\sum_t x_{mn}^{kt} \leq z_{mk}^h \quad m < n \tag{9}$$

$$\sum_k x_{mn}^{kt} \leq z_{nt}^h \quad m < n \tag{10}$$

$$\sum_v y_{uv}^{kl} = p_u^{kl} \qquad u \neq v \tag{11}$$

$$\sum_{u} y_{uv}^{kl} = p_{v}^{kl} \qquad u \neq v \tag{12}$$

$$y_{uv}^{kl} \leq q_{kl} \qquad u \neq v \tag{13}$$

$$q_{kl} \leq q_{k(l-1)} \qquad l > 1 \tag{14}$$

$$\textstyle\sum_{l} p_{u}^{kl} = z_{uk}^{h} \quad k \neq u \tag{15}$$

$$\sum_{u,v} \gamma_{uv} y_{uv}^{kl} \leq T \tag{16}$$

$$f_{uv}^{klm} \leq y_{uv}^{kl} \qquad u \neq v \tag{17}$$

$$\textstyle\sum_{v} f_{kv}^{klm} - \sum_{v} f_{vk}^{klm} = p_{m}^{kl} \qquad k \neq m, k \neq v \tag{18}$$

$$\sum_{u} f_{um}^{klm} - \sum_{u} f_{mu}^{klm} = p_{m}^{kl} \quad k \neq m, u \neq m \tag{19}$$

$$\textstyle\sum_{u} f_{uv}^{klm} = \sum_{u} f_{vu}^{klm} \qquad k \neq m, u \neq v, v \neq m, v \neq k \tag{20}$$

$$xx_{uv}^{mnkl} \leq y_{uv}^{kl} \qquad u \neq v \qquad m \neq n \tag{21}$$

$$\sum_{v} xx_{mv}^{mnkl} - \sum_{v} xx_{vm}^{mnkl} = p_{m}^{kl} \qquad m \neq n, m \neq k, m \neq v \tag{22}$$

$$\textstyle\sum_{v} xx_{kv}^{kjkl} = p_{n}^{kl} \quad k \neq v, k \neq n \tag{23}$$

$$\sum_{v} xx_{vn}^{mnkl} - \sum_{v} xx_{nv}^{mnkl} = p_{n}^{kl} \qquad m \neq n, n \neq k, n \neq v \tag{24}$$

$$\sum_{v} xx_{vk}^{mkkl} = p_{m}^{kl} \quad k \neq v, k \neq m \tag{25}$$

$$\textstyle\sum_{v} xx_{uv}^{mnkl} = \sum_{v} xx_{vu}^{mnkl} \quad u \neq v, m \neq u, n \neq u, k \neq u \tag{26}$$

$$z_{mk}^{h} \in \{0,1\} \; \forall\, m, k \in N \tag{27}$$

$$q_{kl} \in \{0,1\} \; \forall\, k \in N, l \in V \tag{28}$$

$$p_{v}^{kl} \in \{0,1\} \; \forall\, v, k \in N, l \in V, k \neq v \tag{29}$$

$$y_{uv}^{kl} \in \{0,1\} \; \forall\, u, v, k \in N, l \in V, u \neq v \tag{30}$$

$$x_{mn}^{kt} \geq 0 \;\; \forall\, m, n, k, t \in N \; m < n \tag{31}$$

$$f_{uv}^{klm} \geq 0 \;\; \forall\, u, v, k, m \in N, l \in V, u \neq v, k \neq m, k \neq v, k \neq u \tag{32}$$

$$xx_{uv}^{mnkl} \geq 0 \;\; \forall\, m, n, k, u, v \in N, u \neq v, \; k \neq m, k \neq n \tag{33}$$

As we have assumed, the model is a single allocation, constraint (7) assures that each spoke facility is allocated to one hub. Constraint (8) implies that the spoke facility is allocated to the hub if that hub is installed. Constraints (9) and (10) guarantee that the flow between *m* as an origin facility and *n* as a destination facility is routed through hub *k* and *t* in this order if node m is allocated to hub *k* and node *n* is allocated to hub *t*. Constraints (11) and (12) ensure that an arc *uv* must arrive and leave a node, if a vehicle of hub is assigned to service that node. Constraint (13) guarantees that a vehicle l of hub *k* can use an arc *uv*, if this vehicle is first assigned to the hub. Constraint (14) is for assigning vehicles to the node. Constraint (15) assures that when a node is assigned to the hub, then a vehicle must serve it. Constraint (16) ensures that a maximum allowable time for each vehicle is met. Constraints (17-20) are responsible for sub-tour elimination. Constraint (22-26) are responsible for determining which arcs are passed by vehicle l of hub k to serve flow from node m to node n. By defining this variable xx_{uv}^{mnkl}, the actual distance in a local tour is calculated, so the transportation cost in the local tour is real. Constraint (27-33) are non-negativity and integrality constraints.

2.3. Integral model

In this method, both spoke facility and hubs will be located and the routes are determined simultaneously. In this problem, the objective function will be:

$$\min \sum_k F_k^h . z_{kk}^h + \sum_{l,u,v,k} cc_{uv} . y_{uv}^{kl} + \sum_{k,l} q_{kl} . c_l + \sum_{\substack{m,n,k,t \\ m<n \\ k\neq t}} ((\sum_{i,j} w_{ij} \frac{u_{ijmn}.y_m^s y_n^s}{\sum_{uv} u_{ijuv} y_u^s y_v^s}) c_{pt} + (\sum_{i,j} w_{ij} \frac{u_{ijmn}.y_m^s y_n^s}{\sum_{uv} u_{ijuv} y_u^s y_v^s}) c_{tp}). \alpha . x_{mn}^{kt} + \sum_{u,v,m,n,k,l} \left(\sum_{i,j} w_{ij} \frac{u_{ijmn}.y_m^s y_n^s}{\sum_{uv} u_{ijuv} y_u^s y_v^s} \right) . xx_{uv}^{mnkl} . d_{uv} . c + \sum_m F_m^S y_m^s + B * \left(\sum_{i,j} W_{ij} - \sum_{m,n} \sum_{i,j} w_{ij} \frac{u_{ijmn} y_m^s y_n^s}{\sum_{u,v} u_{ijuv} y_u^s y_v^s} \right) \tag{34}$$

Constraints:

$$\sum_k z_{mk}^h = y_m^s \quad for\ all\ m \tag{35}$$

$$y_m^s \epsilon \{0,1\} \ \forall\, m \tag{36}$$

and constraint (8-33)

In the objective function above, the first element is the upgrade cost for a facility to become a hub; the second element is the cost of using link in local tour. The third element is the cost of vehicle allocation to the hub. The fourth element is the transporting cost in hub network. The fifth element is the transportation cost in local tour. The sixth element is the fixed cost to set up a spoke facility in a node. The seventh element is the penalty fee for the lost demands; which none of the facilities could succeed in satisfying the utility. It means that the utility is zero.

3. Linearization and pre-processing of model

In the above formulation, there are some set of nonlinear constraints. In this section, we have applied some technique to change the model to the linear form. For this purpose, we have applied a similar approach to Aros-Vera et al. (2013) and Khosravi and Jokar (2017). As mentioned above, the elements which behave non-linear in sub problem 01 are:

$$p_{ijmn} = \frac{u_{ijmn} y_m^s y_n^s}{\sum_{u,v} u_{ijuv} y_u^s y_v^s} \tag{37}$$

In this model $0 \leq p_{ijmn} \leq 1$ and could turn linear by following equations:

$$p_{ijmn} \leq y_m^s \ \forall i,j,m,n \tag{38}$$

$$p_{ijmn} \leq y_n^s \ \forall i,j,m,n \tag{39}$$

$$\sum_{m,n} p_{ijmn} + p_{ijab} = 1 \ \forall i,j \tag{40}$$

$$p_{ijmn} \leq \frac{u_{ijmn}}{u_{ijuv}} . p_{ijuv} + 1 - y_{uv}^s \ \forall i,j,m,n,k,l \tag{41}$$

$$y_m^s + y_n^s - 1 \leq y_{mn}^s \ \forall m,n \tag{42}$$

$$0.5 y_m^s + 0.5 y_n^s \geq y_{mn}^s \forall m,n \tag{43}$$

Constraints (38) and (39) ensure that the $p\ (i,j,m,n)$ cannot take any value, if there is not any spoke established on node m or node n. Constraint (40) which can be somehow considered as the “flow constraints" states that the initial demand from starting point "*i*" to final destination "*j*" would be either transported via some O-D pair of (*m, n*), or would be lost (going to dummy points "a" and "b").

Constraint (41) essentially expresses that the transported flow via O-D (*m, n*) is compared to the transported flow via O-D (*u, v*) is proportional to their utilities, which is proven in (44-48). It is obvious that if there is no facility in nodes "*m*" and "*n*", its associated probability [p(i, j, m, n)] would equal zero, and hence this constraint holds true. The same is true of "u" and "v" nodes. In case which there is facility located in "m" , "n, "u, "v", $y_m^s = y_n^s = y_u^s = y_v^s = 1$ thus we would have:

$$p(i,j,m,n) \le \frac{u_{ijmn}}{u_{ijuv}}.p(i,j,u,v) \tag{44}$$

$$p(i,j,u,v) \le \frac{u_{ijuv}}{u_{ijmn}}.p(i,j,m,n) \tag{45}$$

Equivalently:

$$p(i,j,m,n) = \frac{u_{ijmn}}{u_{ijuv}}.p(i,j,u,v) \tag{46}$$

Using Eq. (40), we have the following

$$\sum_{m,n} \frac{u_{ijmn}}{u_{ijuv}}.p(i,j,u,v) + \frac{u_{ijab}}{u_{ijuv}}.p(i,j,u,v) = 1 \tag{47}$$

which can be easily re-written as:

$$prob(i,j,u,v) = \frac{u_{ijuv}}{\sum_{m,n} u_{ijmn}} \quad . \tag{48}$$

As we initially assumed that facilities are in *n*, *m*, *l* and *k*, therefore we can obviously replace (37) it with the set of equations, listed above. In sub-problem 02, the hub is non-capacitated, and each domestic facility must be allocated to just one hub. Therefore x_{mn}^{kt} becomes binary variable. As the whole flow from *m* to *n* routed form one path and the link is non-capacitated, the xx_{uv}^{mnkl} becomes zero-one variable. And finally, as each domestic facility is allocated to one local tour, f_{uv}^{klm} is a binary variable. Therefore, in the integrated model, the non-linearity of the model comes from the logit model and can be linearized like the sub-problem 01 and also the multiplication of one binary variable and restricted variable as follows:

$$Z=\left(\sum_{i,j} w_{ij} \frac{u_{ijmn}.y_m^s y_n^s}{\sum_{uv} u_{ijuv} y_u^s y_v^s}\right).xx_{uv}^{mnkl} = p_{ijmn}.xx_{uv}^{mnkl} \tag{49}$$

and could turn linear by following equations:

$$Z \le M.\, xx_{uv}^{mnkl} \tag{50}$$

$$p_{ijmn} \le Z + M(1 - xx_{uv}^{mnkl}) \tag{51}$$

$$p_{ijmn} \ge Z - M(1 - xx_{uv}^{mnkl}) \tag{52}$$

$$Z \ge 0 \tag{53}$$

The above linear technique can be applied for $\left(\frac{u_{ijmn}.y_m^s y_n^s}{\sum_{uv} u_{ijuv} y_u^s y_v^s}\right).x_{mn}^{kt}$

Several properties of the optimal solution of the above formulation and also some additional constraints as valid inequalities can be used to do preprocessing.

Property 01: In any optimal solution for the flow from "*m*" to "*n*", where in the node "*m*", there is a hub facility, $x_{mn}^{kt} = 0$ where $m \ne k$. This means that if the origin facility is a hub by itself, then the flow from this node is not routed from the other two hub nodes in the optimal solution. This property is also applicable for the destination facility. In other words, $x_{mn}^{kt} = 0 \; where \; n \neq t$ if there is hub facility in node "n". This property is the result of triangle inequality which is valid for distance in the hub network.

Property 02: We can substitute the large number "*M*" with the parameter of the model. We can replace *M* by number "1".

Property 03: If $\gamma_{uv} > \frac{T}{2}$, then the arc uv is not in the optimal solution. It means, $y_{uv}^{kl} = 0$ for $\forall$k, l. Because the time for passing the arc is more than half of the time-bound, and the triangle inequality is also valid for the time required for passing arc.

Property 04: The previous argument can be further extended and we can have:
If $y_{uv}^{kl} = 0$ for $\forall$k, l then $xx_{uv}^{mnkl} = 0$ for $\forall$k, l, m, n and $f_{uv}^{klm} = 0$ $\forall$k, l, m.

Property 05: As in the local tour, the multiple traveling salesman problems are solved, the nodes in the local tour can be connected by forward path or backward path. It means that:

$$\sum_{k,l}(y_{uv}^{kl} + y_{vu}^{kl}) \leq 1 \quad \forall\, u, v \tag{54}$$

4. Solution procedure

To find the exact solution for the integrated model, the optimization software such as CPLEX can be used. It should be mentioned that solving these problems needs a considerable computation effort and takes a long time even for the small sized problems. Therefore, we have developed a heuristic algorithm to find the near optimal solution in two stages. This heuristic iterates between these two stages by defining the correct distance between origin and destination points by solving the second stage. More details about these two stages are provided below.

4.1. Stage 01: Hub and Spoke Location

In the first stage, we determine the number of spoke facility, spoke location, the number of hub facility, the hub location and allocate the spoke facility to the hub using the optimization model below.

$$\min \sum_{k} F_k^h . z_{kk}^h + \sum_{m,n}(\sum_{i,j} w_{ij} \frac{u_{ijmn}.y_m^s y_n^s}{\sum_{uv} u_{ijuv} y_u^s y_v^s}) rc_{mn} + \sum_{m} F_m^S\, y_m^s + B * \left(\sum_{i,j} W_{ij} - \sum_{m,n}\sum_{i,j} w_{ij} \frac{u_{ijmn} y_m^s y_n^s}{\sum_{u,v} u_{ijuv} y_u^s y_v^s} \right) \tag{55}$$

Constraints: 8, 27, 35, 36

For this network rc_{mn} is the cost of the route from "m" to "n" for each unit of flow. In the first round, it is assumed as $(c * d_{mk} + \alpha * c_{kl} + c * d_{lm})$ where *c* is the unit cost of flow for each unit of distance and d_{mk} is the direct distance from *m* to *n* and α is a discount factor between two hubs and c_{kl} is a unit transportation cost of inter-hub connection. Then in section 3, the model can be simply transformed to the linear form.

4.2. Stage 02: Routing problem

In this stage, the vehicle allocated to each hub and the route of each vehicle is determined. In each local tour, the vehicle starts from the hub and ends at the hub. So we have solved the traveling salesman problem with the constraint on the tour length. In addition, we have an integrated model with the given hub and spoke location and allocation of spoke to hub.

4.3. Iterative steps between two stages

The sequential procedure (defining the hub and spoke location first, routing second) will result in a local optimum solution. In this algorithm, after the first round, we will modify the routing cost based on the correct distance from m to *n* (rc_{mn}). It means that if the flow from *m* to *n*, passes some nodes in its rout, the real distance should be determined (which will be known after solving the routing problem for each hub) and considered in the first stage for the next round. The real cost of each unit transportation equals: $\sum_{\substack{m,n,k,t \\ m<n \\ k\neq t}} c_{mn}^{kt}.\, x_{mn}^{kt} + \sum_{u,v,k,l} xx_{uv}^{mnkl}.\, d_{uv}.\, c$

5. Computational results

The modified model of the "Domestic Facility, Hub Location and routing" was tested in the Turkish network. Turkish city network was piloted for testing and validating of the model's formulation based on the gravity rule. In this section the performance of iterative heuristic is presented. The main required data such as the distance information for 81 cities, cost of the hub node, the travel time between two cities, the fixed link cost and the traffic demand were extracted from "OR-LIBRARY" (2017). Other data which were realistically considered are utility, cost of locating domestic facilities and lost demand penalty. The number of n cities was randomly chosen among 81 cities in Turkish network. The experiments are run on a Lenovo Laptop, with an Intel Core i7-4720HQ processor at 2.60 GHz and 16 GB of RAM under Windows 10 environment. All formulations were coded in C# and solved using the CPLEX 12.6.3 Library. In all experiments the maximum computing time was set to 86,000 seconds (one day).

Table 1
Comparison between optimal solution of integrated model and sequential approach

	Sequential approach				Integrated approach				
N	S	h	Number of vehicles assigned	Satisfied demand	S	h	Number of vehicles assigned	Satisfied demand	% Objective improvement
6	5	2	2	100%	5	2	2	91.05%	7.56%
10	9	3	4	95.86%	8	3	4	92.08%	11.14%
14	10	4	6	90.76%	10	4	6	87.65%	19.87%
16	13	5	7	87.21%	12	4	7	84.96%	22.87%
20	16	6	7	86.14%	14	5	6	73.48%	19.55%
24	20	7	9	78.60%	18	6	8	70.55%	21.95%

The outcome is illustrated in Table 1 where the column indicated as "s" is the number of spoke location, and "h" is the number of hub locations. Objective Improvement Represents the improvement of the results of employing integrated model in comparison to the non-integrated one (sequential one). The non–integrated solution provides an upper bound for the integrated solution;

$$\text{Objective improvement}=\frac{|integrated-nonintegrated\ objective|}{non\ integrated}\times 100$$

As we set our maximum runtime to 86400 seconds (1 day), we were not able to solve instances with more than 7 nodes within that time period using the integrated model in CPLEX solver. The results of 5, 6, and 7 nodes show that the gap between the heuristic and the CPLEX solver solution which equals to zero. Alpha is assumed as 0.5 and lost demand cost is equal to 0.8 for each value of n in Table 1. The data in this table shows that the integrated approach provides better solutions compared with the sequential approach due to finding the global optimum solutions. On average, the solutions found by heuristic are 18% lower than those found by sequential approach. In sequential approach, regardless of the costs attributed to the hub location, vehicle assignment, and routing, the answer of sub-problem 1 is extracted and then sub-problem 2 is solved. As in sub-problem 1, the optimized solution just trades off between lost demand costs and spoke location cost, we have seen that the number of spoke location in sequential approach is equal or more than the number of spokes in the integrated one. As the percentage of lost demand just depends on the number and location of spokes, the percentage of satisfied demand is decreased in the integrated model in comparison with the sequential one in most cases. This is also the same for 6 nodes network which has an equal number of spoke in the sequential and the integrated model. The percentage of the satisfied demand affirms this trend: we observe lower satisfied demand percentage in an integrated model. This is because of change in the location of spoke nodes.

Table 2
The Results of different interhub transportation cost coefficient

	Alpha=0.2				Alpha=0.5				Alpha=0.8			
N	S	h	Number of vehicles assigned	Satisfied demand	S	h	Number of vehicles assigned	Satisfied demand	S	h	Number of vehicles assigned	Satisfied demand
6	6	2	2	100%	5	2	2	91.05%	5	2	2	91.05%
10	8	3	4	100%	8	3	4	92.08%	8	3	4	92.08%
14	11	4	6	93.79%	10	4	6	87.65%	10	4	6	87.65%
16	12	4	7	84.96%	12	4	7	84.96%	11	4	6	81.94%
20	14	5	6	76.47%	14	5	6	73.48%	14	4	6	70.47%
24	18	7	8	74.94%	18	6	8	70.55%	17	6	8	63.46%

The results of Table 2 show that with increasing the cost coefficient between hubs, the willingness to use hub is decreasing and the amount of satisfied demand is also decreased. For example for the case of 20 node network with 0.2 alpha, the solution has 5 hubs and with 0.8 alpha, only 4 hubs installed. Moreover, the percentage of satisfied demand is decreased even with equal number of the spoke. This is because of changing the location of spokes and increasing the transportation cost.

6. Conclusion

It was introduced a realistic formulation on the "Many to Many Hub and Spoke Location Routing Problem Based on The Gravity Rule" in the present paper. Assumptions as every traveller is served by the nearest facility were relaxed. Instead, traveller's patronage was basically divided among all facilities by applying The Gravity Rule. The problem is very close to the problems such as P-Median location, single allocation hub location and the multi-depot vehicle routing problem. Apparently, all of these models are difficult problems. To linearize the model, linearization techniques were applied and also some tightening rules were proposed in order to further compact the model providing a tighter formulation to help for the solution procedure. As in many hub location problems, the model would not be solved to reach a global optimum especially for the large size problems (it is NP-hard), two approaches (sequential and integrated) were proposed, and computational results were presented. The heuristic for solving the integrated problem is composed of two stages. In the first stage the spoke and hub location and allocation of spoke to the hub were determined and in the second stage, the routing problem for each hub was solved. Results from the second stage to first stage are attained by updating the distance between two nodes based on the real route. The outcome of the computational process on a set of Turkish network problems seems outstanding.

Further research may address how to solve the problem by considering hub congestion or flow based cost coefficient between hubs. Moreover, future research can benefit from examining the solution space as a plane instead of discrete point as a networks.

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