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EOQ model for deteriorating items with exponential demand rate and shortages

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A B S T R A C T

In this paper, we develop an inventory model for deteriorating items with exponential time dependent demand rate. In this proposed model, shortages are allowed and partially backlogged. The model is explained with the help of numerical examples. We also show that total cost function is convex with time. Sensitivity analysis of the optimal solution with respect to various parameters of the system is given and results are discussed in details. Taylor's series expansion is used for finding closed form optimal solution. Some results have been also obtained.

1. Introduction

Sufficient work has already been accomplished by large number of authors in different areas of real life problems for controlling inventory. The main objective of inventory management is to minimize the inventory carrying cost. Therefore, it is very important to determine the optimal stock and optimal time of replenishment of inventory to meet the future demand. The one of the important concerns of the management is to decide when and how much to order so that the total cost associated with the inventory system should be minimum. Deterioration plays an important role in the study of inventory system. Most of the items deteriorate over time. Hence the effect of deterioration cannot be ignored while determining the optimal inventory policy of that type of products.

In classical inventory models the demand rate is assumed to be a constant. But demand for physical goods may be time-dependent, price dependent and stock dependent. An EOQ model with constant demand rate under the conditions of permissible delay in payments has been developed by Goyal (1985). In his model Goyal ignored the difference between the selling price and purchase cost. Goyal's model was extended by Dave (1985) by assuming the fact that the selling price is necessarily higher than its purchase cost. Aggarwal and Jaggi (1995) considered the inventory model with an

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exponential deterioration rate under the conditions of permissible delay in payments. Jamal et al. (2000) then further generalized the model to allow for shortages. Hwang and Shinn (1997) developed the model for determining the retailer's optimal price and lot size simultaneously when the supplier permits delay in payments for an order of a product whose demand rate is a function of price elasticity.

Teng (2002) considered that the selling price is not equal to the purchase price to modify Goyal's model (1985). Huang (2007) investigated the retailer's optimal replenishment policy under permissible delay in payments. Tripathy and Mishra (2011) developed an inventory model on ordering policy for linear deteriorating items for declining demand with permissible delay in payments. Chang et al. (2003) have suggested a model under the conditions that supplier offers trade credit to the buyer if the order quantity is greater than or equal to a pre-determined quantity. Other many related articles can be found by Ouyang et al. (2008), Chang et al. (2003), Chung and Huang (2009) and Teng et al. (2005). In most of the inventory models, holding cost is known as constant, but holding cost may not always be constant. Several researchers like Vander Veen (1967), Goh (1994), Muhlemaan and Valtis Spanopoulos (1980), Weiss (1982) and Naddor (1966) were generalized EOQ models describing various functions of holding cost.

The characteristic of all of the above research papers is that the unsatisfied demand due to shortages is completely backlogged. However in reality, demands for foods, medicines, raw materials etc. are usually lost during the shortage period. Montgomery et al. (1973) studied both deterministic and stochastic demand inventory models with a mixture of backorder and lot sales. Park (1982) modified and established the solution of Mak (1987) by incorporating a uniform replenishment rate to determine the optimal production inventory control policies. Papachristos and Skouri (2002) developed a partially backlogged inventory model in which the backlogging rate decreases exponentially as the waiting time increases.

Teng et al. (2003) then extended the fraction of unsatisfied demand back ordered to any decreasing function of the waiting time up to the next replenishment. Teng and Yang (2004) generalized the partial backlogging EOQ model to allow for time-varying purchase cost. Yang (2005) made a comparison among various partial backlogging inventory lot size models for deteriorating items on the basis of maximum profit. Teng et al. (2007) compared two pricing and lot sizing model for deteriorating items with shortages. Yang et al. (2010) developed an inventory model under inflation for deteriorating items with stock-dependent consumption rate and partial backlogging shortages. Dye et al. (2007) developed inventory and pricing strategies for deteriorating items with shortages. Skouri et al. (2011) proposed an inventory model with general ramp type demand rate, constant deterioration rate, partial backlogging of unsatisfied demand and conditions of permissible delay in payments. Sana (2010) considered that the demand rate is price dependent, the deterioration rate is taken to be time proportion and shortages are allowed. Other related articles on inventory system with partial backlogging and shortages have been performed by Hou (2006), Jagga et al. (2006), Patra et al. (2010), Lin (2012), etc.

In this paper, we develop an EOQ model in which (1) shortages are partial backlogging; (2) demand rate is exponential time-dependent. Several results have been obtained. An optimal solution of the total cost is discussed for various partial backlogging issues. Finally, the sensitivity analysis is given to validate the proposed model. The rest of the paper is organized as follows: section 2 provides assumptions and notations, Mathematical formulation is given in section 3, section 4 indicates the optimal solution, section 5 given some results based on optimal solution, numerical examples are given in section 6 followed by sensitivity analysis is in section 7 and finally conclusion and future research directions are given in section 8.

2. Assumptions and notations

The following assumptions are being made throughout the manuscript:

1. The demand rate $D \equiv D(t)$ of an item is time dependent i.e. $D(t) = \lambda_0 e^{\alpha t}$, where λ_0 and α are constants such that $\lambda_0 > 0$, $0 < \alpha < 1$.
2. The lead time is zero.
3. The replenishment rate is infinite and instantaneous.
4. Shortages are allowed and partially backlogged.
5. The deterioration rate is constant i.e. θ is constant and $0 < \theta < 1$.

In addition the following notations are used throughout the paper

$D \equiv D(t) = \lambda_0 e^{\alpha t}$, demand rate of the items where $\lambda_0 > 0$, $0 < \alpha < 1$.

T :Cycle time

T_1: The time at which $I(t) = 0$

A : Set up cost

c_h : The inventory holding cost per unit per unit of time

c_s : The shortage cost per unit per unit of time

c_d : The cost of each deteriorated items

θ : The constant deterioration rate

$I(t)$: The on hand inventory level at time t over the period $[0, T]$

$C(T_1, T)$: The total cost per year

DC : The deterioration cost

HC : The holding cost

SC : The shortage cost.

3. Mathematical Formulation

The inventory level varies with time due to both demands and deterioration of material. At t = T_1 the inventory level achieves zero, after which shortages are allowed during the time interval (T_1 , T) and all of the demand during shortages period (T_1, T) are partially backlogged. The inventory levels of the model are given by the following differential equation

$$\frac{dI(t)}{dt} + \theta I(t) = -\lambda_0 e^{\alpha t}, 0 \le t \le T_1 \tag{1}$$

$$\frac{dI(t)}{dt} = -\lambda_0, T_1 \le t \le T \tag{2}$$

The solution of Eq. (1) and Eq. (2) with the condition $I(T_1) = 0$ is given by

$$I(t)=\frac{\lambda_0}{(\alpha+\theta)}\left\{e^{(\alpha+\theta)T_1-\theta t}-e^{\alpha t}\right\}, 0\le t\le T_1 \tag{3}$$

and

$$I(t)=\lambda_0(T_1-t), T_1\le t\le T \tag{4}$$

The amount of deteriorated items during $[0,T_1]$ is given by

$$\frac{\lambda_0}{(\alpha+\theta)}\left\{e^{(\alpha+\theta)T_1}-1\right\}-\frac{\lambda_0}{\alpha}\left\{e^{\alpha T_1}-1\right\}. \tag{5}$$

The deterioration cost is given by

$$DC=\lambda_0 c_d\left[\frac{\{e^{(\alpha+\theta)T_1}-1\}}{(\alpha+\theta)}-\frac{(e^{\alpha T_1}-1)}{\alpha}\right] \tag{6}$$

The holding cost during $[0,T_1]$ is given by

$$HC=\frac{\lambda_0 c_h}{(\alpha+\theta)}\left\{\frac{e^{\alpha T_1}(e^{\theta T_1}-1)}{\theta}-\frac{(e^{\alpha T_1}-1)}{\alpha}\right\} \tag{7}$$

The shortage cost during $[T_1,T]$ is given by

$$SC=\frac{c_s\lambda_0}{2}(T-T_1)^2 \tag{8}$$

Therefore, the total cost per year is given by

$$\begin{aligned}C(T_1,T)=\frac{1}{T}\Bigg[&\frac{c_h\lambda_0}{(\alpha+\theta)}\left\{\frac{e^{\alpha T_1}}{\theta}(e^{\theta T_1}-1)-\frac{1}{\alpha}(e^{\alpha T_1}-1)\right\}+c_d\lambda_0\left\{\frac{1}{(\alpha+\theta)}(e^{(\alpha+\theta)T_1}-1)-\frac{1}{\alpha}(e^{\alpha T_1}-1)\right\}\\&+\frac{c_s\lambda_0}{2}(T-T_1)^2+2A\Bigg].\end{aligned} \tag{9}$$

4. Optimal solution procedures

The truncated Taylor's series is used for finding closed form optimal solution in exponential terms i.e. $e^{\alpha T_1}=1+\alpha T_1+\frac{(\alpha T_1)^2}{2!}+.......etc.$ 3rd and higher powers are neglected. This expansion is valid for $\alpha T_1<1$. Using truncated Taylor's series expansion in (9), we get

$$C(T_1,T)=\frac{\lambda_0(c_h+\theta c_d+c_s)T_1^2}{2T}+\frac{c_s\lambda_0 T}{2}-c_s\lambda_0 T_1+\frac{2A}{T} \tag{10}$$

The optimal solution is obtained by taking the 1st and 2nd partial derivatives of (10) with respect to T_1 and T respectively, we get

$$\frac{\partial C(T_1,T)}{\partial T_1}=\frac{\lambda_0(c_h+\theta c_d+c_s)T_1}{T}-c_s\lambda_0 \tag{11}$$

$$\frac{\partial^2 C(T_1,T)}{\partial T_1^2}=\frac{\lambda_0(c_h+\theta c_d+c_s)}{T} \tag{12}$$

$$\frac{\partial C(T_1,T)}{\partial T}=-\frac{\lambda_0(c_h+\theta c_d+c_s)T_1^2}{2T^2}+\frac{c_s\lambda_0}{2}-\frac{2A}{T^2} \tag{13}$$

$$\frac{\partial^2 C(T_1,T)}{\partial T^2}=\frac{\lambda_0(c_h+\theta c_d+c_s)T_1^2}{T^3}+\frac{4A}{T^3} \tag{14}$$

$$\frac{\partial^2 C(T_1,T)}{\partial T\partial T_1}=-\frac{\lambda_0(c_h+\theta c_d+c_s)T_1}{T^2} \tag{15}$$

Since $\left(\frac{\partial^2 C(T_1,T)}{\partial T_1^2}\right).\left(\frac{\partial^2 C(T_1,T)}{\partial T^2}\right)-\left(\frac{\partial^2 C(T_1,T)}{\partial T_1\partial T}\right)^2>0, \frac{\partial^2 C(T_1,T)}{\partial T_1^2}>0$ and $\frac{\partial^2 C(T_1,T)}{\partial T^2}>0$, hence optimal solution obtained on solving (11) and (13) simultaneously will be minimum.

Thus optimal (minimum) solution of $T=T^*$ and $T_1=T_1^*$ is obtained on solving $\frac{\partial C(T_1,T)}{\partial T_1}=0$ and $\frac{\partial C(T_1,T)}{\partial T}=0$ simultaneously, we obtain

$$(c_h+\theta c_d+c_s)T_1-c_sT=0 \tag{16}$$

$$\lambda_0(c_h+\theta c_d+c_s)T_1^2-\lambda_0c_sT^2+4A=0 \tag{17}$$

5. Results

Based on the above equations we obtain the following results:

Result 1. Optimal T_1 is an increasing function of T .

Proof: Differentiating Eq. (16) with respect to T , we obtain

$$\frac{dT_1}{dT}=\frac{c_s}{(c_h+\theta c_d+c_s)}=c(say) \tag{18}$$

Since T_1,T,c_s,θ,c_h and c_d are positive. Hence $\frac{dT_1}{dT}=c>0$. Again differentiating Eq. (17) with respect to T , we get

$$\frac{dT_1}{dT}=\frac{c_sT}{(c_h+\theta c_d+c_s)T_1}=c_1(say) \tag{19}$$

From Eq. (19), we obtain $\frac{dT_1}{dT}=c_1>0$. Hence optimal T_1 is an increasing function of T .

Result 2. $C(T_1,T)$ is convex in T_1 and T both.

Proof: Since $\frac{\partial^2 C(T_1,T)}{\partial T_1^2} > 0, \frac{\partial^2 C(T_1,T)}{\partial T^2} > 0$ and $\left(\frac{\partial^2 C(T_1,T)}{\partial T_1^2}\right).\left(\frac{\partial^2 C(T_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 C(T_1,T)}{\partial T_1 \partial T}\right)^2 > 0$. Hence $C(T_1,T)$ is convex in T_1 and T respectively.

6. Numerical Example

We provide the numerical example to illustrate the inventory model with the following data: the demand D = 1000 for t = 0 i.e. initial demand, the deterioration rate θ = 0.1, the holding cost c_h = 1.0 per item per unit of time, the deterioration cost c_d = 2.5 per item; the shortage cost c_s = 3.5 per item per unit of time; the set up cost A = 20; then we obtain T = 0.294715, T_1 = 0.217159 and $C(T_1,T)$ = 271.448357 which validate $T > T_1$.

7. Sensitivity analysis

We have performed sensitivity analysis by changing parameters c_h, A, c_d, λ_0 and keeping the remaining parameters at their original values. The original values are taken from above example. The corresponding changes in the cycle time T, time T_1 at which I(T_1) = 0 and total cost per year are exhibited in the following Tables 1 to 5.

Table 1
Variation in c_h keeping all the parameters same as in numerical example

c_h	T	T_1	$C(T_1,T)$
1	0.294715	0.217159	271.448357
2	0.241687	0.147114	331.0063706
3	0.217882	0.112976	367.1713698
4	0.204158	0.092201	391.8525068
5	0.19518	0.078072	409.8780306
6	0.188831	0.067786	423.6592729
7	0.184097	0.059939	434.554035
8	0.180428	0.053744	443.391187
9	0.177499	0.048725	450.7075048
10	0.175106	0.044572	456.8667998

Table 2
Variation in A keeping all the parameters same as in numerical example

A	T	T_1	$C(T_1,T)$
25	0.329502	0.242791	303.4884893
30	0.360951	0.265964	332.4549831
35	0.389872	0.287274	359.0924232
40	0.41679	0.307109	383.885948
45	0.442073	0.325738	407.1725355
50	0.465986	0.343358	429.1975376
55	0.48873	0.360117	450.1461751
60	0.510462	0.37613	470.162346
65	0.531306	0.391488	489.3604849
70	0.551362	0.406267	507.8333751
75	0.570714	0.420526	525.657483
80	0.589431	0.434317	542.896714

Table 3
Variation in c_d keeping all the parameters same as in numerical example

c_d	T	T_1	$C(T_1,T)$
5	0.276026	0.193218	289.8275349
6	0.269921	0.18524	296.3834295
7	0.264416	0.177972	302.5532375
8	0.259425	0.171319	308.3737013
9	0.254877	0.165198	313.8765976
10	0.250713	0.159545	319.0896141

Table 4
Variation in λ_0 keeping all the parameters same as in numerical example

λ_0	T	T_1	$C(T_1,T)$
1100	0.281	0.207053	284.6974387
1200	0.269037	0.198238	297.3567767
1300	0.258482	0.190461	309.4987459
1400	0.24908	0.183533	321.1820274
1500	0.240634	0.177309	332.4549831
1600	0.232993	0.171679	343.3580301
1700	0.226036	0.166553	353.9253564
1800	0.219668	0.161861	364.1861872
1900	0.213809	0.157543	374.1657387
2000	0.208395	0.153554	383.885948

Table 5
Variation in c_s keeping all the parameters same as in numerical example

c_s	T	T_1	$C(T_1,T)$
4.5	0.285968	0.223801	279.7514425
5.5	0.28026	0.22836	285.4496129
6.5	0.276238	0.231684	289.6048476
7.5	0.273252	0.234216	292.7700219
8.5	0.270946	0.236209	295.2617266
9.5	0.269112	0.23782	297.2744406
10	0.268328	0.238514	298.142397
11	0.266970	0.239728	299.6596709
12	0.265832	0.240754	300.9419176
13	0.264866	0.241632	302.0398487
14	0.264035	0.242392	302.9905588
15	0.263312	0.243057	303.8218101

The observations found from the Table 1 to Table 5 are summed up as follows:

From Table 1 it can be easily seen that:

- Increase in the values of holding cost c_h results, decrease in the values of T and T_1 but increase in the values of total cost $C(T_1,T)$ keeping all parameters same.

- From Table 2 it can be easily seen that:

Increase in the values of ordering cost A results, increase in the values of T , T_1 and total cost $C(T_1,T)$, keeping all parameters same.

- From Table 3 it can be easily seen that:

Increase in the values of deterioration cost c_d results, decrease in the values of T and T_1 but increase in the values of total cost $C(T_1,T)$ keeping all parameters same.

- From Table 4 it can be easily seen that:

Increase in the values of λ_0 results, decrease in the values of T and T_1 but increase in the values of total cost $C(T_1,T)$ keeping all parameters same.

- From Table 5 it can be easily seen that:
- Increase in the values of shortage cost c_s results, slight decrease in the values of T and slight increase in the values of T_1 but increase in the values of total cost $C(T_1,T)$ keeping all parameters same.

8. Conclusion and future research

In most of the deterministic EOQ model with shortages for permissible items, demand is either constant or induced by pricing decision. In this paper demand rate is taken as exponential time dependent. We have shown that the total cost associated with the inventory system is a convex function of the time. We also have some important results based on optimal solution. From managerial point of view, sensitivity analysis is quite sensitive with respect to variation of parameters.

The proposed model can be extended in several ways. For example, we may extend the model by adding pricing strategy in to consideration. Also, we could extend the model in to stochastic demand. Finally, we could generalize the model to allow inflation, trade credits or others.

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