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A multi-objective robust optimization model for the capacitated P-hub location problem under uncertainty

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A B S T R A C T

Uncertainty plays an important role on many engineering problems and there is a growing interest in having reliable solutions especially for problems with sensitive parameters. The paper presents a robust optimization (RO) model for multi-objective operation of capacitated P-hub location problems (MCpHLP) under uncertainty set. There are, at least, two parameters in any P-hub problems, which are under uncertainty. The first one is associated with demand and the second one is the amount of time required to process commodities. We present a scenario based robust optimization technique, where these two items are considered under various scenario and a RO is implemented to find reliable solutions. The implementation of the proposed RO model is demonstrated for an example using weighting method.

1. Introduction

Hubs are special facilities that are serving as switching in transportation and multistage distribution systems. Hub location problem is concerned with finding the location of hubs and allocating demand points to each hub to route the required traffic between an origin-destination pair. Hubs can be defined as particular facilities in the role of intermediates for distribution systems. By routing and organizing the traffic between each origin-destination pairs (according to a given problem), hubs lead to reduce time, cost, and to improve other parameters.

Models developed on hub location problems are mostly applied to certain set. O'kelly (1987) presented the first recognized mathematical formulation for a hub location problem by studying an airline passenger networks. His formulation was considered with the single allocation p-median allocation problem. Research was followed by a variety of studies. Campbell (1994) developed the first integer linear programming formulations for single allocation p-median problems. Thereafter, hub location problems under certainty set have been broadly investigated. Location problems under uncertainty were first investigated by Ermoliev and Leonardi (1982) who developed some location problem models by formulating uncertainty and solved the resulted problems using uncertainty

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programming devices. Louveaux (1986) reviewed existed uncertain location problems models where all the facility location problems were considered in the first step of decision-making and distribution pattern was regarded as the second step.

Among studies conducted on hub location problems under uncertainty set, only five valid cases have been published. The first article addressed the hub location under uncertainty was presented by (Marianov & Serra, 2003). He used the M/D/c queuing models with a capacity constraint for a plane on landing. The model dealt with hub location optimization in airline networks. Later, (Mohammadi et al., 2011) proposed a model similar to the one used by Marianov and Serra (2003). The difference is that a capacity constraint is added to the model and the M/M/c queuing model is applied. Yang and

Ta-Hui (2009) developed a model for air traffic demand forecasting. The stochastic programming model was introduced for hub location problems in air traffic and flight path programming when the volume of demands varied over seasons. In the same year, another stochastic p-median model was introduced, which minimizes the peak hour travel time by using random constraints to reach guaranteed service level. The problem formulation assumes travel times in a stochastic process with normal distribution. Contreras et al. (2011) studied hub location problem models without capacity constraint. In his study, models with uncertain transportation demands or costs are investigated

To the best of our knowledge, among studies conducted on robust optimization hub location problems, there is only one paper has been published. Huang Jia (2009) presented a robust model for hub location to minimize sum of transportation costs without considering capacity constraints and the resulted problem was solved by multi-objective genetic algorithm.

The current paper is organized as the follow; section 2 gives a brief history of the robust optimization applied here. Section 3 describes and formulates the problem and finally, a particular example is analyzed and solved in section 4.

2. Robust optimization

In this paper, the framework originally developed by Mulvey and Ruszczynsk (1995) is used for the robust optimization to handle uncertainty using different scenario planning. The framework consists of two robustness approaches: solution robustness and model robustness. The first means that the solution for all scenarios must be approximate to the optimum solution, while the latter refers to feasibility of solution for all scenarios. However, no solution, feasible and optimum, could be generally obtained under any scenario. By the concept of multi-criteria decision-making (MCDM), therefore, solution robustness and model robustness can balance. Feng and Rakesh (2010) developed the LP model including random parameters, as below:

Min $c^T x + d^T y$

subject to

$Ax = b,$

$Bx + Cy = e,$

$x, y \geq 0$

where x is the decision variable vector, y is the control variable vector, and B, C, and e are the random values. Let S=1,2,…s be a set of different scenarios for values of the random parameters, and each scenario has a probability value of p^s, ($\sum_s p^s=1$). Note that the model could be infeasible per any scenario of s. Therefore, δ^S is defined as the feasible value. So, if the model were feasible, then δ^S would be equal to zero. Otherwise, it finds a positive value. Hence, the robust optimization model will be given as below:

$Min\ \sigma(x, y^1, \dots, y^s) + \omega\rho(\delta^1, \dots, \delta^s)$

subject to

$Ax = b,$

$B^s x + C^s y^s + \delta^s = e^s \qquad \forall s \in S,$

$x \geq 0, y^s \geq 0, \delta^s \geq 0, \qquad \forall s \in S$

The first part of the objective function considers the solution robustness, and the second part concerns the model robustness.

Mulvey and Ruszczynsk (1995) defined the robust optimization model for the first part of the objective function as:

$$\sigma(0) = \sum_{s \in S} p^s \psi^s + \lambda \sum_{s \epsilon S} p^s \, (\psi^s - \sum_{s' \epsilon S} p^{s'} \psi^{s'})^2$$

Where λ is the weight value allocated to the solution variances. The less sensitive the solution is against the data variances under different scenarios, the higher values for λ. Yu and Li (2000) converted the above quadratic equation into an absolute value and by some modifications developed it as below:

$$Min \sum_{s \in S} p^s \psi^s + \lambda \sum_{s \epsilon S} p^s \left[\left(\psi^s - \sum_{s' \epsilon S} p^{s'} \psi^{s'} \right) + 2\theta^s \right]$$

subject to

$$\psi^s - \sum_{s \epsilon S} p^s \psi^s + \theta^s \geq 0, \qquad \forall s \in S,$$

$$\theta^s \geq 0, \qquad \forall s \in S,$$

The second part of the objective function is associated with the model robustness, composes the penalties applied in the control constraints. Here, we use the coefficient ω as the weight to balance two parts of the objective function. Therefore, the objective function can be presented as:

$$min \sum_{s \in S} p^s \psi^s + \lambda \sum_{s \epsilon S} p^s \left[\left(\psi^s - \sum_{s' \epsilon S} p^{s'} \psi^{s'} \right) + 2\theta^s \right] + \omega \sum_{s \epsilon S} p^s \, \delta^s$$

3. Modeling

3.1 Multi-objective Capacitated p-hub Location Problem under uncertainty (MCpHLP-s)

The idea for uncertainty model, developed here, is exploited from a model proposed by Yand and Ta-Hui (2009). Assuming that there is a given number of airports (n), and also, some volumes of demand for commodities between two airports (D_{ij}). The number of P-hub location, chosen among the present airports, should be established to handle the distribution system. The traveling commodities from a specific origin to a specific destination can utmost go through two hubs. A capacity (U_k) has been defined for each hub.

Commodities are to be processed in each hub with the required time (T_{kl}). Obviously, if commodities travel to their destination by a route with no hub, then the time value is equal to zero, if they go through one hub, $T_{kk} = T_k$, and if they go through two hubs, $T_{kl} = T_k + T_l$. The distance traversed from the origin to the destination, (d_{iklj}), is equal to the sum of distances from the origin to a hub, from that hub to another, and from the latter to the destination,($d_{ik} + d_{kl} + d_{lj}$) , all are the inputs for the problem.

In this model, we assume two parameters; i.e. the demand for commodities between origin-destination pairs (D_{ij}) and the processing time for each hub (T_k) under uncertainty and in scenarios. Other parameters associated with these two uncertain cases are also defined as scenarios. Parameters applied in the model are summarized in Table 1 and the model is formulated as below:

MCpHLP:

$$\min \sum_{k=1}^{n} F_k Z_K + \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}^{s}\, d_{ij} C_{ij} x_{ij}^{s} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} D_{ij}^{s}\, d_{iklj}^{s} C_{iklj}^{s} x_{iklj}^{s} \tag{1}$$

$$\min \max_{i,j}\left(d_{ij} x_{ij}^{s} + \sum_{k=1}^{n}\sum_{l=1}^{n} d_{iklj}^{s} x_{iklj}^{s} \right) \tag{2}$$

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} T_{kl}^{s}\, D_{ij}^{s} x_{iklj}^{s} + \sum_{k=1}^{n} P_k Z_{\,k} \tag{3}$$

subject to

$$\sum_{k=1}^{n} Z_k = p \tag{4}$$

$$\sum_{k=1}^{n}\sum_{l=1}^{n} x_{iklj}^{s} + x_{ij}^{s} = 1 \qquad \forall i,j \;\; i \neq j \;\; D_{ij}^{s} \neq 0, \tag{5}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} D_{ij}^{s} x_{iklj}^{s} \leq U_k Z_k \qquad \forall k \quad i \neq j \tag{6}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} D_{ij}^{s} x_{ilkj}^{s} \leq U_k Z_k \qquad \forall k \quad i \neq j \tag{7}$$

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} \left(x_{iklj}^{s} + x_{ilkj}^{s} \right) - \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ikkj}^{s} \right] \leq M Z_k \quad \forall k \;\; i \neq j \tag{8}$$

$$M\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} \left(x_{iklj}^{s} + x_{ilkj}^{s} \right) - \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ikkj}^{s} \right] \geq Z_k \quad \forall k \;\; i \neq j \tag{9}$$

$$z_k, x_{ij}^s, x_{iklj}^s \in \{0,1\} \qquad \forall i,j,k,l \quad i \neq j \tag{10}$$

Table 1
Notations of model

variable	Description
M	A big positive number
p	Maximum number of hubs which can be established
n	number of nodes in network
β^s	Maximum of distance that be traversed in scenario s
U_k	capacity of hub at node k
F_k	fixed cost of establishing a hub at node k
T_k^s	time hub k takes to process one unit of flow in scenario s
P_k	fixed time to initiate the service at hub k
D_{ij}^s	demand from location i to location j in scenario s
d_{ij}	distance between node i and node j
C_{ij}	the unit transportation cost for the non-stop service between i and j (per each distance unit)
C_{iklj}^s	the unit transportation cost for hub-connected service from i to j and transshipped at hubs k and l in scenario s(per each distance unit)
Z_k	equal 1 if a hub located at node k and otherwise 0
x_{ij}^s	equal 1 if the demand is transported through the non-stop path i–j for scenario s, otherwise 0
x_{iklj}^s	equal 1 if the demand is transported from i to j and transshipped at hubs k and t for scenario s, otherwise 0

The objective function (1) minimized the sum of fixed costs for establishing hubs and of transporting commodities costs. The objective function (2) minimizes the maximum distance traversed. The objective function (3) minimizes the total time values spent for processing commodities, and also for preparation established hubs. Constraint (4) forces us to establish p-hub. Constrain (5) is to ensure the travel of commodities from the origin to the destination. Constraints (6) and (7) are related to the capacity. Constraint (8) indicated that if there is no hub in the node k, then the node must not perform as a hub. Constraint (9) makes the hub necessary to go through when a hub placed on the node k. Constraint (10) defines the problem decision variables.

The above is a nonlinear model, because the objective function (2) is the MiniMax. To make a linear model, the objective function (2) is replaced by the function (11), also the constraint (12) is added to the problem:

$$\min \beta^s \tag{11}$$

$$\beta^s \geq d_{ij}x_{ij}^s + \sum_{\substack{k=1 \\ \neq j}}^{n} \sum_{l=1}^{n} d_{iklj}^s x_{iklj}^s \qquad \forall i,j \quad i \tag{12}$$

3.2 *Robust Optimization Formulation*

In this section, the model MC*p*HLP, proposed in section 3-1, is developed using Mulvey's robust optimization methodology when uncertain parameters are a discontinued scenario. For simplicity, the objective functions are first abbreviated as below:

$$TC^s(transfer\ costs) = \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}^s\, d_{ij} C_{ij} x_{ij}^s + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} D_{ij}^s\, d_{iklj}^s C_{iklj}^s x_{iklj}^s$$

$$FC^s(fix\ costs) = \sum_{k=1}^{n} F_k Z_k$$

$$PT^s(processing\ times) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} T_{kl}^s\, D_{ij}^s x_{iklj}^s$$

$$ST^s(setup\ times) = \sum_{k=1}^{n} P_k Z_k$$

According to the above definitions, the robust optimization model is formulated as:

$$Min\ Z_1 = \sum_s p^s\,(TC^s + FC^s) + \lambda_1 \sum_s p^s \left[(TC^s + FC^s) - \sum_{s'} p^{s'}\left(TC^{s'} + FC^{s'}\right) + 2\theta_1^s \right] + \omega \sum_{s,i,j} p^s\, \delta_{ij}^s\,, \tag{13}$$

$$Min\ Z_2 = \sum_s p^s\,(\beta^s) + \lambda_2 \sum_s p^s \left[(\beta^s) - \sum_{s'} p^{s'}\left(\beta^{s'}\right) + 2\theta_2^s \right], \tag{14}$$

$$Min\ Z_3 = \sum_s p^s\,(PT^s + ST^s) + \lambda_3 \sum_s p^s \left[(PT^s + ST^s) - \sum_{s'} p^{s'}\left(PT^{s'} + ST^{s'}\right) + 2\theta_3^s \right], \tag{15}$$

subject to

$$(TC^s + FC^s) - \sum_s p^s\,(TC^s + FC^s) + \theta_1^s \ge 0, \quad \forall s, \tag{16}$$

$$(\beta^s) - \sum_s p^s\,(\beta^s) + \theta_2^s \ge 0, \quad \forall s, \tag{17}$$

$$(PT^s + ST^s) - \sum_s p^s\,(PT^s + ST^s) + \theta_3^s \ge 0, \quad \forall s, \tag{18}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} \left(D_{ij}^s - \delta_{ij}^s\right) x_{iklj}^s \le U_k Z_k \quad \forall k \quad i \ne j \tag{19}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} \left(D_{ij}^s - \delta_{ij}^s\right) x_{ilkj}^s \le U_k Z_k \quad \forall k \quad i \ne j \tag{20}$$

$$\theta_1^s, \theta_2^s, \theta_3^s, \delta_{ij}^s \ge 0 \quad \forall s, i, j \tag{21}$$

Constraints (4), (5), (8), (9), (10) and (12).

The first and second parts of Eq. (13), Eq. (14) and Eq. (15) represent the mean and variance for the objective functions. The third part of Eq. (13) indicates the amount of model robustness with respect to uncertainty of the constraints Eq. (19) and Eq. (20) under each scenario. Constraints (16-18) are applied to make the model linear as defined. The constraints (19) and (20), the control constraints, are defined the same as the constraints (6) and (7). The difference is that δ_{ij}^s would be a positive value when scenario obtains infeasible solution. Otherwise, $\delta_{ij}^{s} = 0$. Furthermore, Constraint (21) defines non-zero variables.

4. A Given case solution

4.1 Solution Process

The robust optimization model, presented in the previous section, is a multi-objective mixed integer programming. Moreover, all three objective functions are in full contradiction. Therefore, using the weighting method, a popular approach to solve multi-objective models, we can convert the problem into alterative with a single objective function. The objective functions, however, do not have similarly scaled, we first normalize them as follows,

$$Z_i^{norm} = \frac{Z_i - Z_i^*}{Z_i^*}, \tag{22}$$

where, Z_i^* is the ideal value for each objective function. For the proposed optimization model, three objective functions are replaced with the Eq. (23), leading the problem to a single objective function:

$$min\, Z_3 = [\alpha_1 Z_1^{norm} + \alpha_2 Z_2^{norm} + \alpha_3 Z_3^{norm}], \tag{23}$$

where $0 \leq \alpha_i \leq 1$ and $\sum_i \alpha_i = 1$ are the weight coefficients for elements of the objective function given in Eq. (23), determined by the decision maker. The resultant single-objective model (MIP) can be easily solved by different linear model solution software, like Lingo and Gams.

4.2 Experiment

An airline network is assumed with three scenarios; low, middle and high. There are four airports in the network, and we are forced to establish two hubs among them. The fixed costs of hub establishment, amounts of time for hub preparation, and the maximum capacity of each node -when chosen as a hub – are listed in Table 2. Distances between two nodes are presented in Table 3. The volume of demands between two nodes under each scenario and the amount of processing time for commodities in each hub under all scenarios are indicated in Table 4.

Table 2
The input data of the example

Title	Value
The number of nodes	4
Number of hubs which should be established	2
The unit transportation cost for the non-stop service (per each km)	7.5$
The unit transportation costs for hub-connected service (per each km) (high/middle/low)	(3.5$,3$,2.5$)
Probability sets (high/middle/low)	(0.3,0.4,0.3)
(F_1,F_2,F_3,F_4)(million dollar)	(3600,3600,3600,3600)
(P_1,P_2,P_3,P_4)(day)	(12,9,16,15)
(U_1,U_2,U_3,U_4)(number of air cargo)	(50,50,50)

Table 3
Distances between nodes (kms)

nodes	1	2	3	4
1	0	600	450	800
2	-	0	700	300
3	-	-	0	750
4	-	-	-	0

Table 4
Demands and processing times in each scenario

Scenarios	Nodes	1	2	3	4	T_k(day/cargo)
Low	1	0	0	9	22	0.004
	2	0	0	0	18	0.001
	3	8	0	0	2	0.0025
	4	35	2	0	0	0.003
Middle	1	0	4	15	30	0.0045
	2	2	0	8	20	0.0012
	3	10	0	0	5	0.0028
	4	40	5	4	0	0.0033
High	1	0	10	24	35	0.0048
	2	12	0	6	11	0.0015
	3	12	3	0	6	0.003
	4	47	9	5	0	0.0036

The modeling and solution processes of the above problem were performed by the software Lingo in a PC with Core2duo 2.00 GHz CPU and 4 GB of RAM with $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\omega = 300$. The results are given in Table 5. The value of ω will impose significant effect on solutions. If $\omega = 0$, for example, then the maximum value of δ_{ij}^{s} would be obtained. In this case, the average costs reach to their minimum values. The result shows that 2 hubs can be established in airplanes 2 and 3. The average amount of construction and transportation costs is 531,795$; the average maximum distance traversed is 1,360 Km; the average total processing time is 25.11 day; and the average of the sum of the values of δ_{ij}^{s} is 139. Table 5 represents the routes established by one or two hubs under each scenario. The rest, not represented here, has been established without a travel through hubs.

Table 5
Routs with hub/hubs

	Low	Middle	High	
Route	2→ 3→ 1	1→ 3 →2→ 4	1→ 2 →3	4 →2→ 3
	4 →2 →3	1 →3→ 2	1→ 3→ 4	4 →2→ 3 →1
		2→ 3→ 4	2 →3→ 1	
		4 →2→ 1	3 →2→ 1	
		4→ 2 →3	3 →2→ 4	

As stated, the value of objective functions and the amount of δ_{ij}^{s} are affected by ω. Such effect on the present model can be displayed as Fig. 1 and 2. Increasing ω will lead to an increase in the value of objective functions, while the amount of δ_{ij}^{s} shows a decrease.

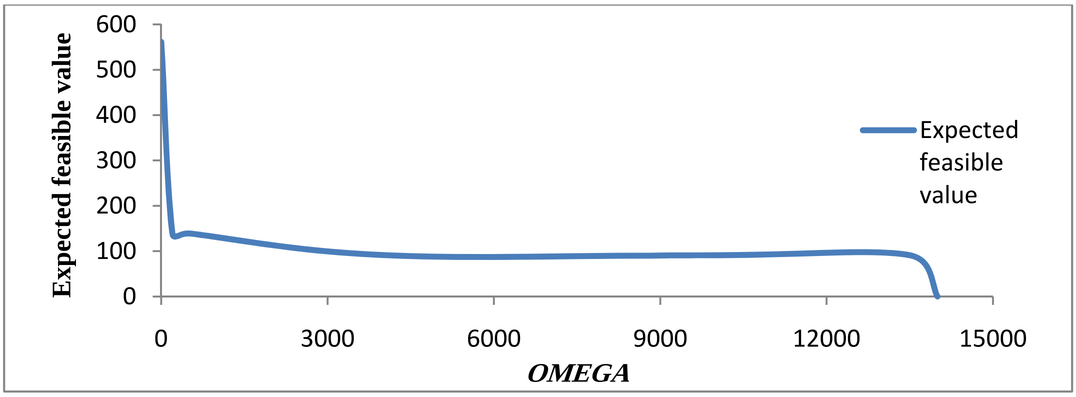

Fig. 1. Trade-off between model robustness and expected feasible value

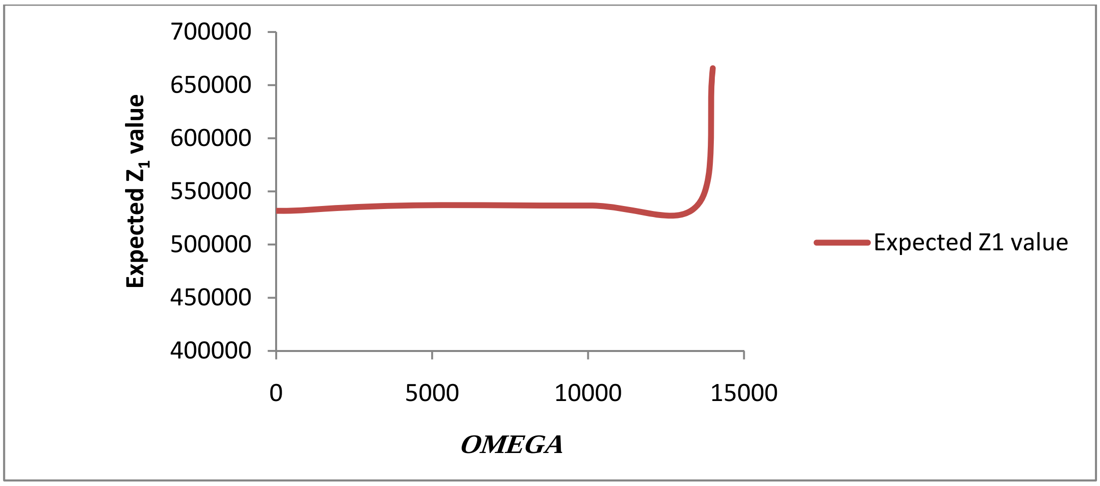

Fig. 2. Trade-off between model robustness and expected Z_1 value

5. Conclusion

The article developed a robust optimization model for multi-objective operation of capacitated P-hub location problems where three objective functions were minimized, simultaneously, including the sum of costs, the maximum distance traversed and the total processing times. To solve it, the robust multi-objective model was converted into a single-objective problem and the weighting method was applied. The volume of demand, processing time and the related costs were presented as different scenarios. The advantage here is that the model is close to real conditions. The solution robustness and model robustness both can be provided by the robust optimization approach simultaneously.

Experimental result indicated that the model robustness increased, but the solution robustness decreased. However, choosing the best ω with trade-off between these may put the decision maker in ideal conditions.

For future studies, goal programming can be hires to solve the model. Moreover, it may be possible to apply Meta-heuristic Innovative Algorithms for the solution methods of the large-scale problems.

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