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Effect of inflation on EOQ model with multivariate demand and partial backlogging and carbon tax policy

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A B S T R A C T

The concept of green inventory systems is very important for economic growth and development in the era of sustainable development. There is a special need for green inventory systems to identify and manage perishable products since spoilage and deterioration can lead to significant losses of items, which negatively affect the satisfaction of consumers. As perishable products decay continuously (such as vegetables, fruits, milk, juices, frozen foods, baked foods), their demand is adversely affected as well as customers' purchasing decisions. The more realistic assumption is a price-sensitive demand. As well as deterioration rates, perishable products have an expiration date-dependent deterioration rate. Further, inventory holding, and the deterioration of perishables contribute significantly to carbon emissions when operating the inventory system. A carbon tax policy is more flexible and effective when it comes to reducing carbon emissions due to its environmental conscious nature. We develop two sustainable inventory procedures for perishable items based on a practical scenario in which the buyer has a limited storeroom. So, to achieve sustainability goals, a model for inventory management for perishable products based on expiration dates is presented in this paper. We distinguish between two inventory schemes: (i) one that allows shortages and fractional backlogs, and (ii) one that does not allow shortage. In both schemes, both the decay rate and demand function show an upward trend against storage time. Since the decay rate increases with storage time, it is assumed that the cost of storing items is linearly related to storage time. numerical examples along with a real-life case study are presented to validate the inventory schemes after several decision-making findings have been derived.

1. Introduction

In making inventory models, demand plays a crucial role. It varies depending on different factors such as time, quality, promotional offers, etc. Consumers are attracted to large displays in stores (Das et al., 2020). Additionally, the product's price is directly related to its consumption level. Dey et al. (2019) investigated whether rebates, stock levels, and price influenced demand. The researchers compared the effect of static and dynamic rebates. In the case of price-sensitive demand, Chang (2013) revisits Burwell's contributions and notes the quantity and freight discounts. According to Ouyng et al. (2008), deterioration occurs under stock-dependent demand, a scenario that includes all unit quantity discounts. Jaggi et al. (2017) established the ordering policy for perishable products, assuming that the selling price depended on the consumption rate. In some nations and international organizations, the carbon cap-and-trade system is utilized as a market-based mechanism to reduce and mitigate carbon dioxide emissions. Below this law, a government agency allocates a predetermined amount of carbon emissions (carbon cap) to a company, and the company can promote or purchase its surplus or more carbon emission permits in a carbon trading market inclusive of the Eu Union Emissions buying and selling the device (European ETS). With the growth of environmental consciousness, an increasing number of customers have a significant incentive to acquire eco-friendly items (Agatz et al., 2012; Thgersen et al., 2012; Stiglic et al., 2015). Many ethical businesses engage in greener technology to minimize carbon emissions as a result of government and consumer demands under

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the cap-and-trade law (Drake et al., 2016). If a product that a customer requires is not now available in the company's inventory, the customer, known as a lost customer, either travels to another location or sets a backorder for that item to meet demand.

To begin with, Zangwill (1966) developed a multi-period production scheduling model with the backlog. Montgomery et al. (1973) developed inventory models that included backorders and lost sales (1972). Rosenberg (1979) provided the study of a partial backlog lot-size model. Mak (1987) finds the best production-inventory control strategies for an inventory system. Inventory shortages were tolerated and partially backlogged. Wee (1993) proposed an economical manufacturing lot size model for degrading products with partial backorder (1993). Abad (1996) developed a generalized model of dynamic pricing and lot-sizing for perishable commodities. Shortages were permitted, and demand was partially met. Sharma and Sadiwala addressed the impact of missed sales on composite lot size (1997). Wee (1993) designed a deteriorating inventory model with a quantity discount, price, and partial backorder (1999). Inventory shortages were tolerated and thought to be primarily backlogged. Ouyang and Chuang (2001) introduced a stochastic continuous review inventory model with variable lead time and partial backorders to account for the realities of unpredictable backorders. Teng et al. proposed an EOQ model for deteriorating products with time-varying demand and partial backlog (2003). Ghosh and Chaudhuri suggested an EOQ model with time-varying degradation and linear time-varying demand across a finite time horizon (2005). Inventory shortages were tolerated and partially backlogged, with a waiting time-dependent backlog rate. A two-level supply chain model with partial backordering and approximated Poisson demand was given by Thangam and Uthayakumar (2008). Shukla et al. (2013) developed a decaying item inventory model. This Model assumed demand was an exponential function, and shortages were tolerated. Sana and Goyal (2015) created an economic order quantity model for different lead times, variable purchase costs, and partial backlog based on the lead time, order size, and reorder point. Luis A. et al. (2017) proposed an optimal inventory strategy for partial backlog and power demand patterns. Chakraborty et al. (2018) created a two-warehouse partial backlog inventory model with ramp type demand rate, three-parameter Weibull distribution degradation under inflation, and allowable payment delay (OuYang et al., 2033). Thinakaran et al. (2019) created a survey model for EOQ and EPQ inventory models with partial backorder problems.

Inflation increases the overall prices of goods and services over time. When the general price level rises, each money unit buys fewer products and services. As a result, inflation indicates a loss of money's purchasing power–a loss of real value in the economy's internal medium of exchange and unit of account. Buzacott initially presented inventory models with inflation in 1975, when he developed the first EOQ (Economic Order Quantity) model that considered inflationary impacts. The average yearly cost was minimized in this Model, and an equation for the EOQ was generated by reducing the average annual cost. Under inflationary situations, Bierman and Thomas (1977) proposed the inventory decision policy. Hariga developed an economic study of dynamic inventory models with non-stationary prices and demand (1994). Chen (1998) suggested a Weibull distributed degradation generalized dynamic programming model for inventory items (1998). Yang (2004) created two-warehouse inventory models for degrading products with constant demand rates under inflation (2004). The shortfalls were permitted, and the models were entirely backlogged. Dey et al. (2019) explored the two-stage inventory issue over a finite time horizon with inflation and the time value of money. Yang, Teng, and Chern (2010) created an economic order quantity model that allowed for partial backlog and shortages. Inflationary impacts and the temporal worth of money were taken into account. Sarkar and Majumder (2013) created an integrated vendor buyer supply chain inventory model with a lower set-up cost for suppliers. Under inflation, Sarkar et al. (2014) constructed an EMQ model with price and time-dependent demand. Rao and Rao (2015) developed an EOQ model with a random item lifespan, a Pareto distribution, no shortages, and zero lead time. In a two-warehouse setting, Tiwari et al. (2016) constructed a model on the impact of trade credit and inflation on retailers' ordering practices for non-instantaneously degrading products. An inventory model for deteriorating products with inflation-induced variable demand under two-level partial trade credit was developed by Pramanik and Maiti (2019). A hybrid ABC-GA strategy.

When we store our goods in warehouses, we consider the idea of degrading things. Then, degradation happens based on the inventory held in warehouses, the facilities occupying warehouses, and the sorts of inventory. Ghare and Schrader (1963) established the first inventory model for perishable products based on the idea of degradation. "Definite goods reduce with time by a fraction that may be approximated by a negative exponential function of time," they said. Covert and Philip (1973) developed an inventory model for degrading products based on the Weibull distribution function's rate of deterioration. Dave (1986) proposed an order level inventory model for degrading commodities in which demand for the products fluctuates over time and is considered to occur instantly at the start of each time unit. Bhunia and Maiti (1998) suggested an EOQ model in which the rate of degradation and the rate of demand are both linearly time dependent. Skouri and Papachristos (2003) proposed a continuous review inventory model for degrading products with time-varying demand and partially time-varying backlog for deteriorating items. The production inventory model of a product with fluctuating demand and degradation was proposed by Goyal and Giri (2003). Ghosh and Chakrabarty (2009) developed an EOQ model with two shortage facilities based on the same rate of item degradation for decaying with differing holding costs. Dye et al.(2007) developed an EOQ model for finite horizon single products when demand and degradation are continuous and differentiable functions of price and time considering shortages. Noblesse et al. (2014) investigated lot sizing in a production/inventory context in which lead times are defined by a queuing model that is endogenously related to the orders placed by the inventory model. A multi-echelon supply chain with a limited production rate was given by Ghiami and Williams (2015). Khurana and

Chaudhary (2016) used an economic order quantity model to determine the best selling price and order quantity for items that deteriorate with time. An optimal lot-size strategy for degrading products with stock-dependent demand considering profit maximization was proposed by Valentn Pando, et al. (2018). Chen (2019) discussed spoiling inventory under stock-level, time-varying, and price-dependent demand. Yu, et al. (2020) created an inventory model for a decaying product that considers carbon emissions.

2. Assumptions and Nomenclature

The following technical assumptions are used to develop our proposed inventory models:

(1) The lead time or replenishment rate is negligible or zero.
(2) Based on a linearly price- and stock-dependent demand pattern.

$$D(p, I(t)) = \begin{cases} a - bp + \alpha I(t) & when\ q(t) > 0 \\ a - bp & when\ q(t) \le 0 \end{cases}$$

.

(3) During the period under consideration, there have been no replacements or repairs made to deteriorated products.
(4) Stock-out periods will allow shortages to accumulate of the demand at a partially accumulated rate $[1+\delta(T-t)]^{-1}$, where $(T-t)$ is the waiting time and $\delta > 0$.
(5) Per unit, holding costs increase as time passes that each unit has been stored as well as the cost of buying the unit. Holding costs are comprised of two parts, as per Alfares and Ghaithan (2016): a constant part g and a linearly and continuously increasing part h.

Nomenclature

Notations	Description
A	Replenishment cost per order
A	Constant part of demand rate (a>0)
B	Coefficient of the price in the demand rate (b>0)
C_1	Purchasing cost per unit
P	Selling price per unit
c_s	Shortage cost per unit
L	Opportunity cost per unit
$\theta(\phi)$	Deteriorating rate $0 \le \theta \le 1$
G	Constant part of holding cost, a fraction of unit purchase cost
H	Coefficient of linearly time varying holding cost, as a fraction of unit purchase cost
D	Per unit processing cost of the deteriorated product
I(t)	Inventory level
α	Stock-dependent demand rate parameter ($\alpha > 0$)
δ	Backlogging parameter
R	Maximum number of partially backlogged quantity (for model With shortages)
B	The initial number of inventories (for model with shortages)
Q	The number of order quantity per replenishment cycle
R	Rate of inflation
M	Retailer's trade credit period provided by the supplier
t_1	Time at which the stock reaches zero (for model with shortages)
T	Length of each replenishment cycle (for both models)
i_e	Interest rate earned by the retailer.
i_P	Interest rate paid by the retailer.
A_e	Fixed carbon emissions of per order
C_e	Per unit carbon emission of orders
h_e	Per unit carbon emission for inventory per unit time
λ	Carbon tax under the carbon tax policy (doller/ per unit carbon emission)
$TC_1(T)$	Total cost for model without shortages
$TC_2(T)$	Total cost for model with shortages

3. Model formulation

Consider a case where a retailer buys a deteriorating product from a supplier under a discount environment in which the purchase cost is based on the order quantity and decreases step by step over time. Under the temporary discount scheme, there are two proposed inventory models for a single deteriorating item: 1) no shortages; and 2) a partial shortage due to customer waiting times. Initially, the Model without shortages will be discussed, followed by the Model with partial back-logs of shortages when payment delays are permitted.

3.1 Model without shortages

Assume that Q units of the deteriorating item are started in this case. In the long run, the inventory level gradually decreases due to declining demand and deterioration, eventually reaching zero at one point. After a short time, the replenishment process will be started, and the entire inventory system will be implemented (see Fig. 1). The following differential difference equation can be used to describe the inventory level at any given point in time:

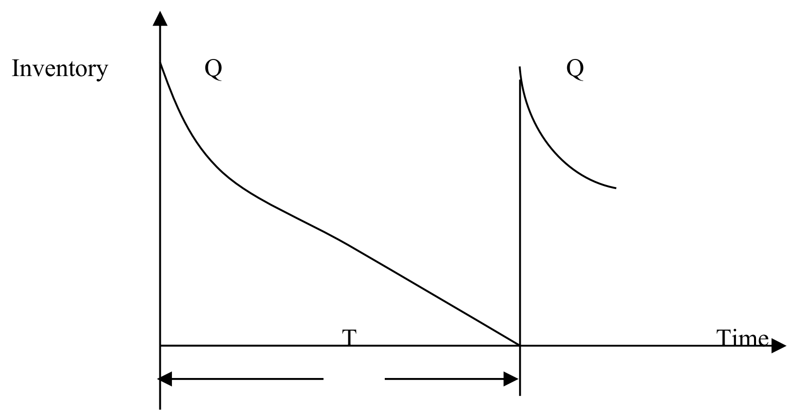

Fig. 1. Graphical presentation of proposed inventory system when non-terminating situation appears

$$I^{'}(t)+\theta(\phi)q(t)=-\left(a-bp+\alpha q(t)\right), \qquad 0\le t\le T \tag{1}$$

With the boundary conditions $I(0)=Q$ and $I(T)=0$
Eq. (1) can be expressed as follows:

$$I^{'}(t)+\mu q(t)=-D, \qquad 0\le t\le T, \tag{2}$$

where $D=a-bp, \mu=\alpha+\theta(\phi).$
According to Eq. (2), whenever there is an inventory level at any given time, it is equal to

$$I(t)=\frac{D}{\mu}\left(e^{\mu(T-t)}-1\right). \tag{3}$$

with $I(0)=Q,$ we get

$$Q=\frac{D}{\mu}\left(e^{\mu T}-1\right). \tag{4}$$

Cost components of this inventory system are as follows:

Sales Revenue cost(SR): $p\int_0^T D\left(p,I(t)\right)e^{-rt}dt$

$$pD\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)$$

Ordering cost (OC): A

Holding cost (HC): $c\int_0^T (g+ht)I(t)e^{-rt}dt$

$$=\frac{cD}{\mu}\left(g\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)$$

Deterioration cost(DC): $d\left(Q-\int_0^T D\left(p,I(t)\right)e^{-rt}dt\right)$

$$d\left(\frac{D}{\mu}\left(e^{\mu T}-1\right)-\frac{(\mu-\alpha)D}{r}\left(1-e^{-rT}\right)-\frac{\alpha D}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)$$

Purchasing cost (PC): $\frac{cD}{\mu}\left(e^{\mu T}-1\right)$.

Total carbon emission in a finite time horizon T is given by

$$TE=A_e+C_eQ+h_e\int_0^T I(t)e^{-rt}dt=A_e+C_e\frac{D}{\mu}\left(e^{\mu T}-1\right)+\frac{h_eD}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)$$

Carbon emission cost:

$$=\lambda\left(A_e+C_e\frac{D}{\mu}\left(e^{\mu T}-1\right)+\frac{h_eD}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)\right)$$

Regarding the permissible delay period M for selling the accounts, there two cases arise:

Case (1): $M \le T$

When the permissible delay period is less than the inventory period .In this case the buyer can use the sale revenue to earn the interest with the rate i_e per unit time and pays the interest with the rate i_p per unit time.

The interest payable per year $=\frac{ci_p}{T}\int_M^T I(t)e^{-rt}dt=\frac{ci_pD}{T\mu}\left(\frac{e^{(T-M)\mu-rM}}{(r+\mu)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{e^{-rM}}{r}\right)$

The interest earned per year $=\frac{pi_e}{T}\int_0^M Dte^{-rt}dt$

$$=\frac{pi_e}{T}D\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]$$

Case (2): $M > T$

When the permissible delay period is greater than the inventory period .In this case, the buyer earns the interest during the period (*0,M*) and pays no interest.

The interest payable per year $=0$

The interest earned per year $=\frac{pi_eD}{T}\left(\int_0^T te^{-rt}dt+(M-T)T\right)$

$$=\frac{pi_eD}{T}\left(\frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rT}(1+Tr)\right)+(M-T)T-\frac{\alpha e^{-rT}}{\mu(\mu+r)^2}\left(T(\mu+r)-1\right)\right)$$

Hence total cost for the inventory

$$TC_1=\begin{cases}TC_{1.1} & if\ M\le T\\ TC_{1.2} & if\ M>T\end{cases}$$

where, $TC_1=\frac{1}{T}\left(SR-OC-HC-PC-IP+IE-\lambda TE\right)$

$$TC_{1.1}=\frac{1}{T}\left[\begin{array}{l}(p-d)D\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)-A-\lambda A_e-\frac{(c+d+\lambda C_e)D}{\mu}\left(e^{\mu T}-1\right)-\\ \frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)\\ -\frac{ci_pD}{\mu}\left(\frac{e^{(T-M)\mu-rM}}{(r+\mu)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{e^{-rM}}{r}\right)+\\ pi_eD\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha e^{\mu T}}{\mu(r+\mu)^2}\left(1-e^{-(\mu+r)M}\left(M(\mu+r)+1\right)\right)\right]\end{array}\right] \tag{5}$$

$$TC_{1.2}=\frac{1}{T}\left[\begin{array}{l}(p-d)D\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)-A-\lambda A_e-\frac{(c+d+\lambda C_e)D}{\mu}\left(e^{\mu T}-1\right)-\\ \frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)\\ +pi_eD\left(\frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rT}(1+Tr)\right)+(M-T)T-\frac{\alpha e^{-rT}}{\mu}\left(\frac{T}{(\mu+r)}-\frac{1}{(\mu+r)^2}\right)\right)\end{array}\right] \tag{6}$$

3.2 Model with partial backlogged shortages

Stock is replenished with Q units of the item at the beginning of a cycle. To satisfy the total accumulated backlog, R units are used. In contrast, the inventory level becomes S. Consumed products will deteriorate constantly during a specified time to meet customers' demands. Due to the demand of customers and the deterioration of the inventory, at time t= t_1 the inventory level drops to zero. Shortages begin to emerge soon after. They accumulate throughout the period and depend on the customers' waiting time. The entire inventory system is repeated now that a new order has been placed. A graphical representation of the whole inventory system that spans the full cyclical length is shown in Figure 2. Governing differential equations can be used to describe the inventory level at any given time;

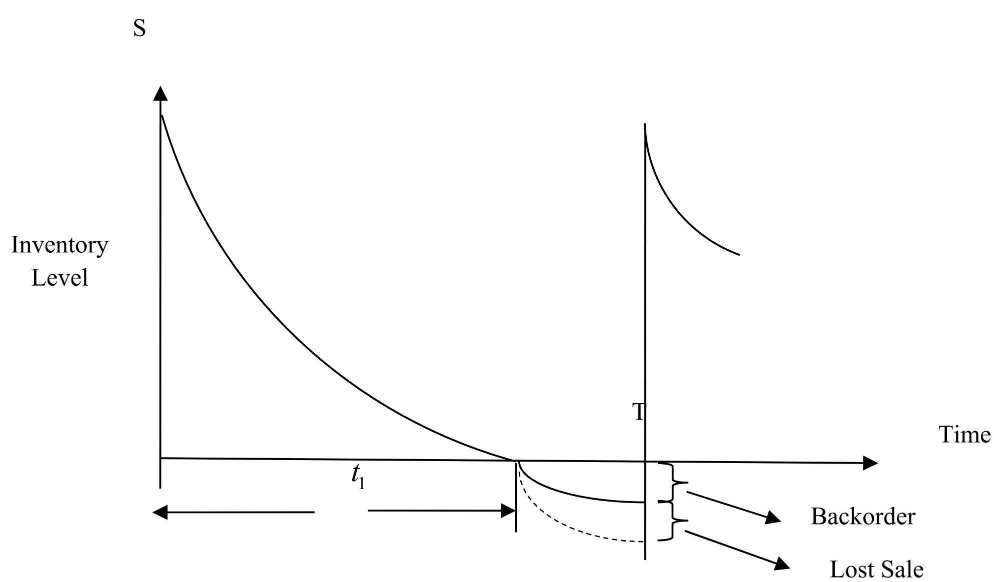

Fig. 2. Pictorial presentation of the proposed model with partial backlogged shortages

$$I_1^{'}(t)+\theta I_1(t)=-[a-bp+\alpha I_1(t)], \qquad 0<t\le t_1 \tag{7}$$

$$I_2^{'}(t)=\frac{-(a-bp)}{1+\delta(T-t)}, \qquad t_1<t\le T \tag{8}$$

with the boundary conditions $I_1(0)=S,\, I_1(t_1)=0$ and $q_2(T)=-R.$

Eq. (7) can be rewritten as

$$I_1^{'}(t)+\mu I_1(t)=-D, \qquad 0<t\le t_1 \tag{9}$$

where $D=a-bp, \mu=\alpha+\theta.$
with condition $I_1(t_1)=0$, $I_1(0)=S$ and $I_2(T)=-R$

$$I_1(t)=\frac{D}{\mu}\left(e^{\mu(t_1-t)}-1\right) \tag{10}$$

and

$$I_2(t)=\frac{D}{\delta}\ln\left(1+\delta(T-t)\right)-R. \tag{11}$$

$$S=\frac{D}{\mu}\left(e^{\mu t_1}-1\right). \tag{12}$$

Again, with the help of continuity of $I(t)$ at $t=t_1$ the maximum shortage level is given by

$$R=\frac{D}{\delta}\ln\left(1+\delta(T-t_1)\right). \tag{13}$$

The total number of ordering quantity is given by

$$Q=\frac{D}{\mu}\left(e^{\mu t_1}-1\right)+\frac{D}{\delta}\ln\left(1+\delta(T-t_1)\right). \tag{14}$$

The profit function of the inventory system has the following components:

Sales Revenue cost (SR): $p\int_0^{t_1} D\left(p,I(t)\right)e^{-rt}dt$

$$pD\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rt_1}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT-(\mu+r)t_1}\right)\right)$$

Ordering cost (OC): A

Deterioration cost(DC): $d\left(Q-\int_0^{t_1} D\left(p,I(t)\right)e^{-rt}dt\right)$

$$dD\left(\frac{1}{\mu}\left(e^{\mu t_1}-1\right)-\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rt_1}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT-(\mu+r)t_1}\right)\right)\right)$$

Holding cost (HC):

$$c\int_0^{t_1}(g+ht)I_1(t)e^{-rt}dt$$

$$=\frac{cD}{\mu}\left(g\left(\frac{e^{\mu t_1}}{(\mu+r)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rt_1}(\mu+2r)}{r^2(\mu+r)^2}+\frac{t_1\mu e^{-rt_1}}{r(\mu+r)}+\frac{e^{\mu t_1}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)$$

Purchasing cost (PC):

$$=cD\left[\frac{1}{\mu}\left(e^{\mu t_1}-1\right)+\frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right]$$

Shortage cost (SC):

$$=-s\int_{t_1}^{T} I_2(t)e^{-rt}dt$$

$$=\frac{-s}{r}\left[De^{-rt_1}(T-t_1)+\frac{(D+Rr)}{r}(e^{-rT}-e^{-rt_1})\right]$$

Lost Sale cost (LSC):

$$=l\int_{t_1}^{T}\left(1-\frac{1}{(1+\delta(T-t))}\right)e^{-rt}dt$$

$$=\frac{lD\delta}{r}\left(e^{-rt_1}\left(T-t_1+\frac{1}{r}\right)-\frac{1}{r}e^{-rT}\right)$$

Total carbon emission in a finite time horizon T is given by

$$TE=A_e+C_eQ+h_e\int_0^T q(t)e^{-rt}dt=A_e+C_eD\left(\frac{1}{\mu}\left(e^{\mu t_1}-1\right)+\frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right)$$

$$+\frac{h_eD}{\mu}\left(\frac{e^{\mu t_1}}{(\mu+r)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{1}{r}\right)$$

Carbon emission cost:

$$CEC=\lambda\left(A_e+C_eD\left(\frac{1}{\mu}\left(e^{\mu t_1}-1\right)+\frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right)+\frac{h_eD}{\mu}\left(\frac{e^{\mu t_1}}{(\mu+r)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{1}{r}\right)\right)$$

(i) Case $0<M\le t_1$

In this case, since the credit period M is smaller than the length of period with positive inventory stock of items, therefore the buyers can use the sale revenue to earn the interest with the rate i_e per unit time.

The interest payable per year

$$=\frac{ci_p}{T}\int_M^{t_1} q_1(t)e^{-rt}dt=\frac{ci_pD}{T\eta}\left(\frac{e^{(t_1-M)\eta-rM}}{(r+\eta)}+\frac{\eta e^{-rt_1}}{r(\eta+r)}-\frac{e^{-rM}}{r}\right)$$

The interest earned par year

$$=\frac{pi_e}{T}\left(\int_0^M D\left(p,I(t)\right)te^{-rt}dt\right)=\frac{pi_eD}{T}\left[\frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rM}(1+rM)\right)+\frac{\alpha e^{-\mu t_1}}{\mu(\mu+r)^2}\left(1-(1+M(\mu+r))e^{-M(\mu+r)}\right)\right]$$

(ii) Case $t_1\le M\le T$

The interest payable per year $=0$

The interest earned per year

$$=\frac{pi_e}{T}\left(\int_0^{t_1} D\left(p,I(t)\right)te^{-rt}dt+\int_{t_1}^{M}\left(\int_0^{t_1}Ddt\right)e^{-rt}dt\right)$$

$$=\frac{pi_eD}{T}\left[\frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rt_1}(1+rt_1)\right)+\frac{\alpha e^{-rt_1}}{\mu(\mu+r)^2}\left(1-(1+t_1(\mu+r))\right)+\frac{t_1}{r}\left(e^{-rM}-e^{-rt_1}\right)\right]$$

(iii) Case $M>T$

The interest earned and paid are similar to case (ii).

Hence total cost

$$TC_2=\begin{cases}TC_{2.1} & if\ 0<M\le t_1\\ TC_{2.2} & if\ t_1\le M\le T\\ TC_{2.3} & if\ M>T\end{cases}$$

$$TC_{2.1}=\frac{1}{T}\begin{bmatrix}(p-d)\left(\frac{D(\mu-\alpha)}{r}\left(1-e^{-rt_1}\right)+\frac{\alpha D}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT-(\mu+r)t_1}\right)\right)-\frac{Dd}{\mu}\left(e^{\mu t_1}-1\right)-A-\lambda A_e\\ -\frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu t_1}}{(\mu+r)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rt_1}(\mu+2r)}{r^2(\mu+r)^2}+\frac{t_1\mu e^{-rt_1}}{r(\mu+r)}+\frac{e^{\mu t_1}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)-\\ (c+\lambda C_e)D\left[\frac{1}{\mu}\left(e^{\mu t_1}-1\right)+\frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right]+\frac{sD}{r}\left[e^{-rt_1}(T-t_1)+\left(1+\frac{r}{\delta}\ln\left(1+\delta(T-t_1)\right)\right)(e^{-rT}-e^{-rt_1})\right]-\\ \frac{lD\delta}{r}\left(e^{-rt_1}\left(T-t_1+\frac{1}{r}\right)-\frac{1}{r}e^{-rT}\right)-\frac{ci_p}{\mu}\left(\frac{e^{(t_1-M)\mu-rM}}{(r+\mu)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{e^{-rM}}{r}\right)+\\ pi_eD\left[\frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rM}(1+rM)\right)+\frac{\alpha e^{-\mu t_1}}{\mu(\mu+r)^2}\left(1-(1+M(\mu+r))e^{-M(\mu+r)}\right)\right]\end{bmatrix}$$

$$TC_{2.2}=TC_{2.3}=\frac{1}{T}\begin{bmatrix}(p-d)\left(\frac{D(\mu-\alpha)}{r}\left(1-e^{-rt_1}\right)+\frac{\alpha D}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT-(\mu+r)t_1}\right)\right)-\frac{d}{\mu}\left(e^{\mu t_1}-1\right)-A-\lambda A_e-\\ \frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu t_1}}{(\mu+r)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rt_1}(\mu+2r)}{r^2(\mu+r)^2}+\frac{t_1\mu e^{-rt_1}}{r(\mu+r)}+\frac{e^{\mu t_1}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)-\\ (c+\lambda C_e)D\left[\frac{1}{\mu}\left(e^{\mu t_1}-1\right)+\frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right]+\frac{sD}{r}\left[e^{-rt_1}(T-t_1)+\frac{(D+Rr)}{r}(e^{-rT}-e^{-rt_1})\right]-\\ \frac{lD\delta}{r}\left(e^{-rt_1}(T-t_1)+\frac{1}{r}\left(e^{-rt_1}-e^{-rT}\right)\right)+\\ pi_eD\left[\frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rt_1}(1+rt_1)\right)+\frac{\alpha e^{-rt_1}}{\mu(\mu+r)^2}(1-(1+t_1(\mu+r)))+\frac{t_1}{r}\left(e^{-rM}-e^{-rt_1}\right)\right]\end{bmatrix}$$

To maximize the retailer's profit per unit of time, the retailer needs to determine the optimal time period t_1^* for positive inventory levels and replenishment T^*. The next section explores the existence and uniqueness of the most suitable answer for each objective function.

4. Theoretical results

For the Model without backlogged shortages, we first show the concavity of the profit function (6). For the Model with partial backlogged needs resulting from customer waiting time (16), we offer the concavity of the total cost function (16). We used some Cambini and Martein (2009) results to examine the concavity of both models. The function of form (A) is derived from the theory of Cambini and Martein (2009) via the theorems 3.2.9 and 3.2.10.

$$\prod(x)=\frac{f(x)}{g(x)},\qquad x\in R^n$$

is (strictly) pseudo-concave if *f (x)* is a negative, differentiable, and (strictly) concave function, but *g(x)* is a positive, differentiable, and concave function. We can explain this result by showing that the profit cost function (6) is a strict pseudo-concave function of T, and we can then find the optimal solution which maximizes the total profit function (6).

4.1. Model without shortages

Lemma 1: Total profit function $TC_{1.1}(p,T)$ is strictly pseudo-concave if

$$-2bi_e\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]<$$
$$2b\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)$$

Proof: From Eq. (6), we define

$$f_{1.1}(T,p)=\begin{bmatrix}(p-d)D\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)-A-\lambda A_e-\frac{(c+d+\lambda C_e)D}{\mu}\left(e^{\mu T}-1\right)-\\ \frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)\\ -\frac{ci_p(a-bp)}{\mu}\left(\frac{e^{(T-M)\mu-rM}}{(r+\mu)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{e^{-rM}}{r}\right)+\\ +pi_e(a-bp)\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]\end{bmatrix}$$

And $g_{1.1}(T)=T>0.$

Therefore, the profit function is $TC_{1.1}(T,p)=\frac{f_{1.1}(T,p)}{g_{1.1}(T,p)}$. Now first and second order derivative w.r.t. to p are

$$\frac{\partial f_{1.1}(T,p)}{\partial p}=\begin{bmatrix}(a-2bp+bd)\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)+\frac{b(c+d+\lambda C_e)}{\mu}\left(e^{\mu T}-1\right)+\\ \frac{bc}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)\\ +\frac{bci_p}{\mu}\left(\frac{e^{(T-M)\mu-rM}}{(r+\mu)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{e^{-rM}}{r}\right)+\\ i_e(a-2bp)\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]\end{bmatrix}$$

$$\frac{\partial^2 f_{1.1}(T,p)}{\partial p^2}=\begin{bmatrix}-2b\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)-\\ i_e 2b\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]\end{bmatrix}$$

First principal minor

$|H_{11}|<0$

If $$\begin{bmatrix}-2b\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)-\\ i_e 2b\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]\end{bmatrix}<0$$

$$-2bi_e\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]<$$

$$2b\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)$$

Lemma 2: Total profit function $TC_{1.1}(p,T)$ is strictly pseudo-concave if

$$\frac{\partial^2 TC_{1.1}(p,T)}{\partial p^2}\frac{\partial^2 TC_{1.1}(p,T)}{\partial T^2}-\left(\frac{\partial^2 TC_{1.1}(p,T)}{\partial T\partial p}\right)^2>0$$

Proof: $$H=\begin{bmatrix}\frac{\partial^2 TC_{1.1}(p,T)}{\partial p^2} & \frac{\partial^2 TC_{1.1}(p,T)}{\partial p\partial T}\\ \frac{\partial^2 TC_{1.1}(p,T)}{\partial p\partial T} & \frac{\partial^2 TC_{1.1}(p,T)}{\partial T^2}\end{bmatrix}.$$

And the corresponding determinate value $|H|>0$,

Now if we apply the rule of determinate the $\frac{\partial^2 TC_{1.1}(p,T)}{\partial p^2}\frac{\partial^2 TC_{1.1}(p,T)}{\partial T^2}-\left(\frac{\partial^2 TC_{1.1}(p,T)}{\partial T\partial p}\right)^2>0$

Now the first- and second-order derivatives of $f_{1.1}(T)$ with respect to T are

$$\frac{\partial f_{1.1}(T,p)}{\partial T}=(p-d)D\left((\mu-\alpha)e^{-rT}+\frac{\alpha}{\mu(\mu+r)}\left(\mu e^{\mu T}+re^{-rT}\right)\right)-(c+d+\lambda C_e)De^{\mu T}$$
$$-cD\left(\frac{(g+\lambda h_e)}{(\mu+r)}(e^{\mu T}-e^{-rT})+h\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{(\mu+2r)e^{-rT}}{r(\mu+r)^2}+\frac{e^{-rT}}{\mu r(\mu+r)}(1-Tr)\right)+\frac{i_p}{(\mu+r)}\left(e^{(T-M)\mu-rM}-e^{-rT}\right)\right)$$
$$-\frac{pi_e\alpha(a-bp)}{(\mu+r)}\left(\frac{e^{\mu T}}{(\mu+r)}(1-e^{-M(\mu+r)})-Me^{(T-M)\mu-rM}\right)$$

$$\frac{\partial^2 f_{1.1}(T,p)}{\partial T^2}=(p-d)D\left(-r(\mu-\alpha)e^{-rT}+\frac{\alpha}{\mu(\mu+r)}\left(\mu^2e^{\mu T}-r^2e^{-rT}\right)\right)-(c+d+\lambda C_e)D\mu e^{\mu T}$$
$$-cD\left(\frac{(g+\lambda h_e)}{(\mu+r)}(\mu e^{\mu T}+re^{-rT})+h\left(\frac{\mu e^{\mu T}}{(\mu+r)^2}+\frac{(\mu+2r)e^{-rT}}{(\mu+r)^2}-\frac{Tre^{-rT}}{\mu(\mu+r)}\right)+\frac{i_p}{(\mu+r)}\left(\mu e^{(T-M)\mu-rM}+re^{-rT}\right)\right)$$
$$-\frac{pi_e\alpha(a-bp)}{(\mu+r)}\left(\frac{\mu e^{\mu T}}{(\mu+r)}(1-e^{-M(\mu+r)})-M\mu e^{(T-M)\mu-rM}\right)$$

Theorem 1.1 The total profit function $TC_{1.1}(p,T)$ is strictly pseudo-concave function with respect to p, T, and hence $TC_{1.1}(p,T)$ archives the global maximum at the point p*, T^*.

Proof: From Eq. (6), we define

$$f_{1.1}(T,p)=\begin{bmatrix}(p-d)D\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)-A-\lambda A_e-\frac{(c+d+\lambda C_e)D}{\mu}\left(e^{\mu T}-1\right)-\\ \frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)\\ -\frac{ci_p(a-bp)}{\mu}\left(\frac{e^{(T-M)\mu-rM}}{(r+\mu)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{e^{-rM}}{r}\right)\\ +pi_e(a-bp)\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]\end{bmatrix}$$

And $g_{1.1}(T)=T>0.$

Therefore, the profit function is $TC_{1.1}(T,p)=\dfrac{f_{1.1}(T,p)}{g_{1.1}(T,p)}.$

From Lemma 1 & 2

$f_{1.1}(p,T)$ is a negative, differentiable, and concave function. Also $g_{1.1}(p,T)=T$ is positive, differentiable, and concave function with respect to T. This implies that the profit function in is a pseudo concave function in T, and that there exists an optimal solution T^*. Thus, the theorem is proved.

Lemma 3: Total profit function $TC_{1.2}(p,T)$ is strictly pseudo-concave if

$$-2bi_e\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rM}}{r}\left(M+\frac{1}{r}\right)\right)+\frac{\alpha}{\mu}\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{e^{(T-M)\mu-rM}}{(r+\mu)}\left(M+\frac{1}{(\mu+r)}\right)\right)\right]<$$
$$2b\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)$$

Proof:

$$f_{1.2}(T,p)=\begin{bmatrix}(p-d)D\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)-A-\lambda A_e-\frac{(c+d+\lambda C_e)D}{\mu}\left(e^{\mu T}-1\right)-\\ \frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)\\ +pi_e(a-bp)\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rT}}{r}\left(T+\frac{1}{r}\right)\right)+(M-T)T+\frac{\alpha e^{-rT}}{\mu}\left(\frac{T}{(\mu+r}-\frac{1}{(r+\mu)^2}\right)\right]\end{bmatrix}$$

And $g_{1.1}(T)=T>0.$

Therefore, the profit function is $TC_{1.2}(T,p)=\dfrac{f_{1.2}(T,p)}{g_{1.2}(T,p)}.$ Now first and second order derivative w.r.t. to p are

$$\frac{\partial f_{1.2}(T,p)}{\partial p}=\begin{bmatrix}(a-2bp+bd)\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)+\frac{b(c+d+\lambda C_e)}{\mu}\left(e^{\mu T}-1\right)+\\ \frac{bc}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)\\ +i_e(a-2bp)\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rT}}{r}\left(T+\frac{1}{r}\right)\right)+(M-T)T+\frac{\alpha e^{-rT}}{\mu}\left(\frac{T}{(\mu+r}-\frac{1}{(r+\mu)^2}\right)\right]\end{bmatrix}$$

$$\frac{\partial^2 f_{1.2}(T,p)}{\partial p^2}=\begin{bmatrix}-2b\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)- \\ i_e 2b\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rT}}{r}\left(T+\frac{1}{r}\right)\right)+(M-T)T+\frac{\alpha e^{-rT}}{\mu}\left(\frac{T}{(\mu+r}-\frac{1}{(r+\mu)^2}\right)\right]\end{bmatrix}$$

First principal minor

$$|H_{11}|<0$$

$$\begin{bmatrix}-2b\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)- \\ i_e 2b\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rT}}{r}\left(T+\frac{1}{r}\right)\right)+(M-T)T+\frac{\alpha e^{-rT}}{\mu}\left(\frac{T}{(\mu+r}-\frac{1}{(r+\mu)^2}\right)\right]\end{bmatrix}<0$$

If $-i_e 2b\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rT}}{r}\left(T+\frac{1}{r}\right)\right)+(M-T)T+\frac{\alpha e^{-rT}}{\mu}\left(\frac{T}{(\mu+r}-\frac{1}{(r+\mu)^2}\right)\right]<$

$$2b\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)$$

Lemma 4: Total profit function $TC_{1.2}(p,T)$ is strictly pseudo-concave if

$$\frac{\partial^2 TC_{1.2}(p,T)}{\partial p^2}\frac{\partial^2 TC_{1.2}(p,T)}{\partial T^2}-\left(\frac{\partial^2 TC_{1.2}(p,T)}{\partial T\partial p}\right)^2>0$$

Proof: $H=\begin{bmatrix}\frac{\partial^2 TC_{1.2}(p,T)}{\partial p^2} & \frac{\partial^2 TC_{1.2}(p,T)}{\partial p\partial T} \\ \frac{\partial^2 TC_{1.2}(p,T)}{\partial p\partial T} & \frac{\partial^2 TC_{1.2}(p,T)}{\partial T^2}\end{bmatrix}.$

And the corresponding determinate value |H|>0,

Now if we apply the rule of determinate the $\frac{\partial^2 TC_{1.2}(p,T)}{\partial p^2}\frac{\partial^2 TC_{1.2}(p,T)}{\partial T^2}-\left(\frac{\partial^2 TC_{1.2}(p,T)}{\partial T\partial p}\right)^2>0$

Now the first- and second-order derivatives of $f_{1.2}(T)$ with respect to T are

$$\frac{\partial f_{1.2}(T,p)}{\partial T}=(p-d)D\left((\mu-\alpha)e^{-rT}+\frac{\alpha}{\mu(\mu+r)}\left(\mu e^{\mu T}+re^{-rT}\right)\right)-(c+d+\lambda C_e)De^{\mu T}$$

$$-cD\left(\frac{(g+\lambda h_e)}{(\mu+r)}(e^{\mu T}-e^{-rT})+h\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{(\mu+2r)e^{-rT}}{r(\mu+r)^2}+\frac{e^{-rT}}{\mu r(\mu+r)}(1-Tr)\right)\right)$$

$$+pi_e(a-bp)\left[\left(1-\frac{\alpha}{\mu}\right)\frac{Te^{-rT}}{r}+(M-2T)+\frac{\alpha re^{-rT}}{\mu}\left(\frac{T(\mu+r)-1}{(\mu+r)^2}\right)+\frac{\alpha e^{-rT}}{\mu(\mu+r)}\right]$$

$$\frac{\partial^2 f_{1.2}(T,p)}{\partial T^2}=(p-d)D\left(-r(\mu-\alpha)e^{-rT}+\frac{\alpha}{\mu(\mu+r)}\left(\mu^2e^{\mu T}-r^2e^{-rT}\right)\right)-(c+d+\lambda C_e)D\mu e^{\mu T}$$

$$-cD\left(\frac{(g+\lambda h_e)}{(\mu+r)}(\mu e^{\mu T}+re^{-rT})+h\left(\frac{\mu e^{\mu T}}{(\mu+r)^2}+\frac{(\mu+2r)e^{-rT}}{(\mu+r)^2}-\frac{(2-Tr)e^{-rT}}{\mu(\mu+r)}\right)\right)$$

$$+pi_e(a-bp)\left(\left(1-\frac{\alpha}{\mu}\right)\frac{(1-rT)}{r}-2+\left(\frac{-\alpha r}{(\mu+r)^2}+\frac{\alpha r(1-rT)}{\mu(\mu+r)}\right)e^{-rT}\right)$$

$$\frac{\partial^2 f_{1.2}(T,p)}{\partial p\partial T}=(a-2bp+bd)\left((\mu-\alpha)e^{-rT}+\frac{\alpha}{\mu(\mu+r)}\left(\mu e^{\mu T}+re^{-rT}\right)\right)+b(c+d+\lambda C_e)e^{\mu T}$$

$$+bc\left(\frac{(g+\lambda h_e)}{(\mu+r)}(e^{\mu T}-e^{-rT})+h\left(\frac{e^{\mu T}}{(\mu+r)^2}-\frac{(\mu+2r)e^{-rT}}{r(\mu+r)^2}+\frac{e^{-rT}}{\mu r(\mu+r)}(1-Tr)\right)\right)$$

$$+i_e(a-2bp)\left[\left(1-\frac{\alpha}{\mu}\right)\frac{Te^{-rT}}{r}+(M-2T)+\frac{\alpha re^{-rT}}{\mu}\left(\frac{T(\mu+r)-1}{(\mu+r)^2}\right)+\frac{\alpha e^{-rT}}{\mu(\mu+r)}\right]$$

Theorem 1.2 The total profit function $TC_{1.2}(T)$ is strictly pseudo-concave function in *T,* under the condition and hence there exists a unique optimal solution T^*.

Proof.

From Eq. (7), we define

$$f_{1.2}(T,p)=\left[\begin{array}{l}(p-d)D\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rT}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT}\right)\right)-A-\lambda A_e-\frac{(c+d+\lambda C_e)D}{\mu}\left(e^{\mu T}-1\right)-\\ \frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu T}}{(\mu+r)}+\frac{\mu e^{-rT}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rT}(\mu+2r)}{r^2(\mu+r)^2}+\frac{Te^{-rT}}{r(\mu+r)}+\frac{e^{\mu T}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)\\ +pi_e(a-bp)\left[\left(1-\frac{\alpha}{\mu}\right)\left(\frac{1}{r^2}-\frac{e^{-rT}}{r}\left(T+\frac{1}{r}\right)\right)+(M-T)T+\frac{\alpha e^{-rT}}{\mu}\left(\frac{T}{(\mu+r}-\frac{1}{(r+\mu)^2}\right)\right]\end{array}\right]$$

And

$$g_{1.2}(T)=T>0$$

Therefore, the profit function is $TC_{1.2}(T,p)=\frac{f_{1.2}(T,p)}{g_{1.2}(T,p)}$.

From Lemma 3 & 4

$f_{1.2}(p,T)$ is a non-negative, differentiable, and concave function. Also $g_{1.2}(p,T)=T$ is positive, differentiable, and concave function with respect to T. This implies that the profit function in is a pseudo concave function in T, and that there exists an optimal solution T^*. Thus, the theorem is proved.

4.2. Model with shortages

Theorem 2.1 The total cost function $TC_{2.1}(T,t_1,p)$ is strictly pseudo- concave function in t_1 , p and T , hence there exists a unique optimal solution.

Proof. For the convenience, let us define the following auxiliary functions:

$$f_{2.1}=\begin{bmatrix} (p-d)\left(\frac{D(\mu-\alpha)}{r}\left(1-e^{-rt_1}\right)+\frac{\alpha D}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT-(\mu+r)t_1}\right)\right)-\frac{Dd}{\mu}\left(e^{\mu t_1}-1\right)-A-\lambda A_e \\ -\frac{cD}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu t_1}}{(\mu+r)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rt_1}(\mu+2r)}{r^2(\mu+r)^2}+\frac{t_1\mu e^{-rt_1}}{r(\mu+r)}+\frac{e^{\mu t_1}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)- \\ (c+\lambda C_e)D\left[\frac{1}{\mu}\left(e^{\mu t_1}-1\right)+\frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right]+\frac{sD}{r}\left[e^{-rt_1}(T-t_1)+\left(\frac{1}{r}+\frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right)(e^{-rT}-e^{-rt_1})\right]- \\ \frac{lD\delta}{r}\left(e^{-rt_1}\left(T-t_1+\frac{1}{r}\right)-\frac{1}{r}e^{-rT}\right)-\frac{ci_p}{\mu}\left(\frac{e^{(t_1-M)\mu-rM}}{(r+\mu)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{e^{-rM}}{r}\right)+ \\ pi_eD\left[\frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rM}(1+rM)\right)+\frac{\alpha e^{-\mu t_1}}{\mu(\mu+r)^2}\left(1-(1+M(\mu+r))e^{-M(\mu+r)}\right)\right] \end{bmatrix}$$

and

$$g_{2.1}(t_1,T,p)=T>0$$

Therefore, the profit function is $TC_{2.1}(t_1,T,p)=\dfrac{f_{2.1}(t_1,T,p)}{g_{2.1}(t_1,T,p)}$.

$$\frac{\partial f_{2.1}}{\partial p}=\begin{bmatrix} (a-bd+2bp)\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rt_1}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT-(\mu+r)t_1}\right)\right)+\frac{bd}{\mu}\left(e^{\mu t_1}-1\right)+ \\ \frac{cb}{\mu}\left((g+\lambda h_e)\left(\frac{e^{\mu t_1}}{(\mu+r)}+\frac{\mu e^{-rt_1}}{r(\mu+r)}-\frac{1}{r}\right)+h\left(\frac{\mu e^{-rt_1}(\mu+2r)}{r^2(\mu+r)^2}+\frac{t_1\mu e^{-rt_1}}{r(\mu+r)}+\frac{e^{\mu t_1}}{(\mu+r)^2}-\frac{1}{r^2}\right)\right)+ \\ (c+\lambda C_e)b\left[\frac{1}{\mu}\left(e^{\mu t_1}-1\right)+\frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right]-\frac{sb}{r}\begin{bmatrix} e^{-rt_1}(T-t_1)+\left(\frac{1}{r}+\frac{1}{\delta}\ln[1+\delta(T-t_1)]\right) \\ (e^{-rT}-e^{-rt_1}) \end{bmatrix}+ \\ \frac{lb\delta}{r}\left(e^{-rt_1}\left(T-t_1+\frac{1}{r}\right)-\frac{1}{r}e^{-rT}\right)+(a-2bp)i_e\begin{bmatrix} \frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rM}(1+rM)\right)+ \\ \frac{\alpha e^{-rt_1}}{\mu(\mu+r)^2}\left(1-(1+M(\mu+r))e^{-M(\mu+r)}\right)-\frac{R}{r}\left(e^{-rM}-1\right) \end{bmatrix} \end{bmatrix}$$

$$\frac{\partial^2 f_{2.1}}{\partial p^2}=\begin{bmatrix} (-2bp)\left(\frac{(\mu-\alpha)}{r}\left(1-e^{-rt_1}\right)+\frac{\alpha}{\mu(\mu+r)}\left(e^{\mu T}-e^{-rT-(\mu+r)t_1}\right)\right) \\ -2bi_e\left[\frac{1}{r^2}\left(1-\frac{\alpha}{\mu}\right)\left(1-e^{-rM}(1+rM)\right)+\frac{\alpha e^{-rt_1}}{\mu(\mu+r)^2}\left(1-(1+M(\mu+r))e^{-M(\mu+r)}\right)-\frac{R}{r}\left(e^{-rM}-1\right)\right] \end{bmatrix}$$

$$\frac{\partial f_{2.1}}{\partial t_1}=\begin{bmatrix}(p-d)\left(D(\mu-\alpha)e^{-rt_1}+\frac{\alpha D}{\mu}e^{-rT-(\mu+r)t_1}\right)-Dde^{\mu t_1}-(c+\lambda C_e)D\left[e^{\mu t_1}-\frac{1}{(1+\delta(T-t_1))}\right]-\\ cD\left((g+\lambda h_e)\left(\frac{e^{\mu t_1}}{(\mu+r)}-\frac{e^{-rt_1}}{(\mu+r)}\right)+h\left(\frac{-e^{-rt_1}(\mu+2r)}{r(\mu+r)^2}-\frac{t_1e^{-rt_1}}{(\mu+r)}+\frac{e^{-rt_1}}{r(\mu+r)}+\frac{e^{\mu t_1}}{(\mu+r)^2}\right)\right)\\ +\frac{sD}{r}\left[-re^{-rt_1}(T-t_1)-e^{-rt_1}-\frac{r}{(1+\delta(T-t_1))}\left(e^{-rT}-e^{-rt_1}\right)-re^{-rt_1}\left(1+\frac{r}{\delta}\log[1+\delta(T-t_1)\right)\right]-\\ \frac{lD\delta}{r}\left(-re^{-rt_1}\left(T-t_1+\frac{1}{r}\right)-e^{-rt_1}\right)-ci_p\left(\frac{e^{(t_1-M)\mu-rM}}{(r+\mu)}-\frac{e^{-rt_1}}{(\mu+r)}\right)+\\ pi_eD\left[\frac{-\alpha e^{-\mu t_1}}{(\mu+r)^2}\left(1-(1+M(\mu+r))e^{-M(\mu+r)}\right)\right]\end{bmatrix}$$

$$\frac{\partial^2 f_{2.1}}{\partial t_1^2}=\begin{bmatrix}(p-d)D\left(-r(\mu-\alpha)e^{-rt_1}-(\mu+r)\frac{\alpha}{\mu}e^{-rT-(\mu+r)t_1}\right)-Dd\mu e^{\mu t_1}-(c+\lambda C_e)D\left[\mu e^{\mu t_1}-\frac{\delta}{(1+\delta(T-t_1))}\right]-\\ cD\left((g+\lambda h_e)\left(\frac{\mu e^{\mu t_1}}{(\mu+r)}+\frac{re^{-rt_1}}{(\mu+r)}\right)+h\left(\frac{e^{-rt_1}(\mu+2r)}{(\mu+r)^2}+\frac{t_1e^{-rt_1}}{(\mu+r)}-\frac{2e^{-rt_1}}{(\mu+r)}+\frac{\mu e^{\mu t_1}}{(\mu+r)^2}\right)\right)+\\ \frac{sD}{r}\left[r^2e^{-rt_1}(T-t_1)+2re^{-rt_1}-\frac{r\delta}{(1+\delta(T-t_1))^2}\left(e^{-rT}-e^{-rt_1}\right)-\frac{2r^2e^{-rt_1}}{1+\delta(T-t_1)}+r^2e^{-rt_1}\left(1+\frac{r}{\delta}\log[1+\delta(T-t_1)\right)\right]-\\ \frac{lD\delta}{r}\left(r^2e^{-rt_1}\left(T-t_1+\frac{1}{r}\right)+2re^{-rt_1}\right)-ci_p\left(\frac{\mu e^{(t_1-M)\mu-rM}}{(r+\mu)}+\frac{re^{-rt_1}}{(\mu+r)}\right)+\\ pi_eD\left[\frac{\alpha\mu e^{-\mu t_1}}{(\mu+r)^2}\left(1-(1+M(\mu+r))e^{-M(\mu+r)}\right)\right]\end{bmatrix}$$

$$\frac{\partial^2 f_{2.1}(.)}{\partial T^2}=-e^{-rT}(a-bp)\delta l-\frac{(e^{-Mr}-1)p(a-bp)\delta i_e}{r(1+\delta(T-t_1))^2}-\frac{(a-bp)\delta c}{(1+T\delta-\delta t_1)^2}$$

$$\frac{\partial^2 f_{2.1}(.)}{\partial T\partial t_1}=D\begin{pmatrix}-\frac{\alpha r}{\mu}(p-d)e^{-rT-(\mu+r)t_1}-(c+\lambda C_e)\frac{\delta}{(1+\delta(T-t_1))^2}+\\ \frac{s}{r}\left(-re^{-rt_1}+\frac{r\delta\left(e^{-rT}-e^{-rt_1}\right)}{(1+\delta(T-t_1))^2}+\frac{r^2\left(e^{-rT}-e^{-rt_1}\right)}{(1+\delta(T-t_1))}\right)+l\delta e^{-rt_1}\end{pmatrix}$$

$$\frac{\partial f_{2.1}(.)}{\partial T}=D\begin{pmatrix}\frac{(p-d)}{\mu(\mu+r)}\left(\mu e^{\mu T}+re^{-rT-(\mu+r)t_1}\right)-(c+\lambda C_e)\frac{1}{(1+\delta(T-t_1))}+\\ \frac{s}{r}\left(e^{-rt_1}+\frac{\left(e^{-rT}-e^{-rt_1}\right)}{(1+\delta(T-t_1))}-re^{-rT}\left(\frac{1}{r}+\frac{1}{\delta}\log[1+\delta(T-t_1)]\right)\right)+l\delta e^{-rt_1}\end{pmatrix}$$

$$\frac{\partial^2 f_{2.1}(.)}{\partial T^2}=D\begin{pmatrix}\frac{(p-d)}{\mu(\mu+r)}\left(\mu^2e^{\mu T}-r^2e^{-rT-(\mu+r)t_1}\right)-(c+\lambda C_e)\frac{\delta}{(1+\delta(T-t_1))^2}+\\ \frac{s}{r}\left(\frac{\left(e^{-rT}-e^{-rt_1}\right)}{(1+\delta(T-t_1))^2}-\frac{re^{-rT}}{(1+\delta(T-t_1))}+\left(re^{-rT}+\frac{r^2e^{-rT}}{\delta}\log[1+\delta(T-t_1)]-\frac{re^{-rT}}{(1+\delta(T-t_1))}\right)\right)\end{pmatrix}$$

$$\frac{\partial^2 f_{2.1}(.)}{\partial p\partial T}=-b\begin{pmatrix}\frac{(p-d)}{\mu(\mu+r)}\left(\mu e^{\mu T}+re^{-rT-(\mu+r)t_1}\right)-(c+\lambda C_e)\frac{1}{(1+\delta(T-t_1))}+\\ \frac{s}{r}\left(e^{-rt_1}+\frac{\left(e^{-rT}-e^{-rt_1}\right)}{(1+\delta(T-t_1))}-re^{-rT}\left(\frac{1}{r}+\frac{1}{\delta}\log[1+\delta(T-t_1)]\right)\right)+l\delta e^{-rt_1}\end{pmatrix}+$$

$$\frac{(a-bp)}{\mu(\mu+r)}\left(\mu e^{\mu T}+re^{-rT-(\mu+r)t_1}\right)$$

$$\frac{\partial f_{2.1}}{\partial p \partial t_1} = \begin{bmatrix} (a-2bp+bd)\left((\mu-\alpha)e^{-rt_1}+\frac{\alpha}{\mu}e^{-rT-(\mu+r)t_1}\right)+bde^{\mu t_1}+(c+\lambda C_e)b\left[e^{\mu t_1}-\frac{1}{(1+\delta(T-t_1))}\right]+ \\ bc\left((g+\lambda h_e)\left(\frac{e^{\mu t_1}}{(\mu+r)}-\frac{e^{-rt_1}}{(\mu+r)}\right)+h\left(\frac{-e^{-rt_1}(\mu+2r)}{r(\mu+r)^2}-\frac{t_1 e^{-rt_1}}{(\mu+r)}+\frac{e^{-rt_1}}{r(\mu+r)}+\frac{e^{\mu t_1}}{(\mu+r)^2}\right)\right) \\ -\frac{sb}{r}\left[-re^{-rt_1}(T-t_1)-e^{-rt_1}-\frac{r}{(1+\delta(T-t_1))}\left(e^{-rT}-e^{-rt_1}\right)-re^{-rt_1}\left(1+\frac{r}{\delta}\log[1+\delta(T-t_1))\right)\right]+ \\ \frac{lb\delta}{r}\left(-re^{-rt_1}\left(T-t_1+\frac{1}{r}\right)-e^{-rt_1}\right)-ci_p\left(\frac{e^{(t_1-M)\mu-rM}}{(r+\mu)}-\frac{e^{-rt_1}}{(\mu+r)}\right)- \\ pi_e b\left[\frac{-\alpha e^{-\mu t_1}}{(\mu+r)^2}\left(1-(1+M(\mu+r))e^{-M(\mu+r)}\right)\right] \end{bmatrix}$$

Therefore, the Hessian matrix for $f_{2.1}(t_1,T)$ is

$$H_{ii} = \begin{bmatrix} \frac{\partial^2 f_{2.1}(.)}{\partial t_1^2} & \frac{\partial^2 f_{2.1}(.)}{\partial t_1 \partial T} & \frac{\partial^2 f_{2.1}(.)}{\partial t_1 \partial p} \\ \frac{\partial^2 f_{2.1}(.)}{\partial T \partial t_1} & \frac{\partial^2 f_{2.1}(.)}{\partial T^2} & \frac{\partial^2 f_{2.1}(.)}{\partial T \partial p} \\ \frac{\partial^2 f_{2.1}(.)}{\partial p \partial t_1} & \frac{\partial^2 f_{2.1}(.)}{\partial p \partial T} & \frac{\partial^2 f_{2.1}(.)}{\partial p^2} \end{bmatrix}$$

Here both the principal minor $|H_{11}| = \frac{\partial^2 f_{2.1}(.)}{\partial t_1^2} > 0$

and $|H_{22}| = \frac{\partial^2 f_{2.1}(.)}{\partial t_1^2}\frac{\partial^2 f_{2.1}(.)}{\partial T^2} - \frac{\partial^2 f_{2.1}(.)}{\partial T \partial t_1}\frac{\partial^2 f_{2.1}(.)}{\partial t_1 \partial T} > 0$

$|H_{33}| > 0$

Since all the principal minors of the Hessian matrix for $f_{2.1}(t_1,T,p)$ have alternate sign, the Hessian matrix is negative definite. Therefore, $f_{2.1}(t_1,T,p)$ is a negative, differentiable, and (strictly) concave function with respect to t_1 and T simultaneously. Moreover, the function $g_{2.1}(t_1,T,p)=T$ is negative, differentiable, and concave function, so our total cost function per unit time $TC_{2.1}(T,t_1,p)$ is pseudo-concave function in t_1 and T, and has only one maximum value.
Similarly, we construct theorem from $T_{2.2}(t_1,T,p)$

5. Numerical illustration

We present two numerical examples for two different cases to illustrate our theoretical results and also to gain managerial insight into our problems.

Example 1. For case of without shortages model

A=\$0.6/Order, a=22, b=0.06,g=\$0.04/unit, h=\$0.1/unit, θ=0.0,r=0.18, M=0.5year, c=\$0.05/unit, α=0.01, η=0.2, i_p =\$0.1/year, i_e=\$0.08/year, $\delta = 0.05$, r=0.01, C_j=0.5, $\lambda = 0.01$, A_e = 0.05, C_e=0.6, $h_e = 0.03$

We get from (6) $T \to 0.298689$ year , $TC_{1.1} \to \$1001.97$ and $p = 183.864$

And from (7) we get T =0.30145year, $TC_{1.2} \to \$987.53$ and $p = 181.764$

Comparing both profit we find that $TC_{1.1}$ is an optimal profit, regarding this optimal selling price is p* = 183.864 and optimal cycle length T*= 0.298689year.

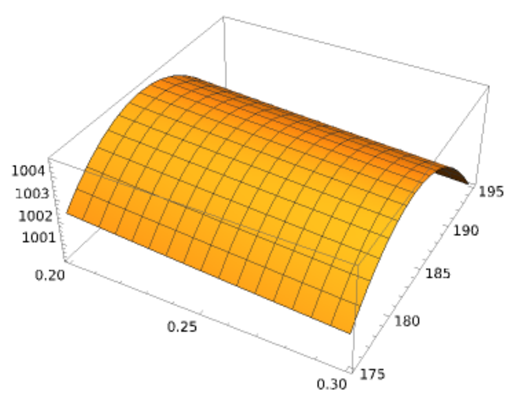

Fig. 3. Behavior of the total profit, selling price and cycle length

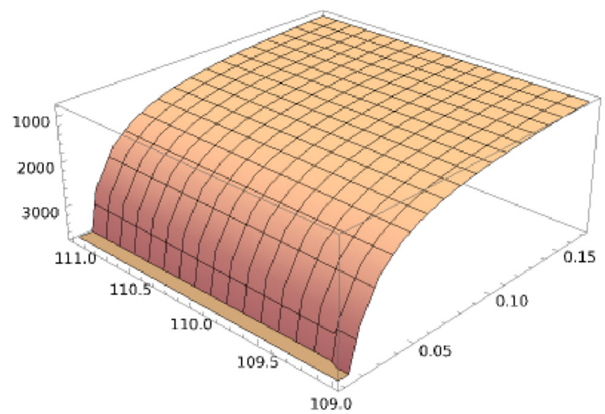

Fig. 4. Behavior of the optimal profit between cycle length and selling price

Example 2. For case of shortages model

The data are same those are used in Example 1. Additionally, we consider that $\delta = 0.02, C_s = 0.4, C_l = 0.5$,for the backlogged shortages.

Solution:

Optimal Total profit $TC_{2.1}$=\$600.39, T=2.03 year t_1= 1.82214 year and p = \$75.695

Optimal Total profit $TC_{2.2}$=\$1069.37, T=0.1401 year, t_1= 0.124538year and p = \$105.772

Comparing every profit we find that optimal profit is $TC^*_{2.2}$ and regarding this selling price and cycle length is optimal. From equation (12), (13), (14) we get S*,R*, and Q*

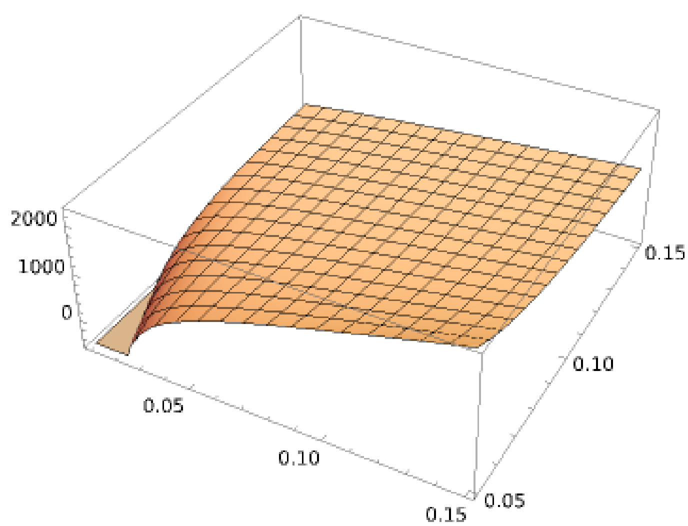

Fig. 5. Behavior of the optimal profit between cycle length and replenishment time

5.1 Sensitivity analysis for without shortage w.r.t. various parameter

Table 1
Sensitivity analysis for variations in holding cost (A)

A	Total Cost ($TC_{1.1}^{*}$)	Cycle Time (T*)	Selling price(p*)
+20%	1001.61	0.34036	183.864
+15%	1001.7	0.330443	183.864
+10%	1001.79	0.320215	183.864
+5%	1001.89	0.309642	183.864
0%	1001.97	0.298689	183.864
-5%	1002.09	0.287318	183.862
-10%	1002.19	0.275461	183.862
-15%	1002.31	0.26307	183.861
-20%	1002.42	0.250053	183.861

Table 2
Sensitivity analysis for carbon emission deterioration rate (C_e)

C_e	Total Cost ($TC_{1.1}^{*}$)	Cycle Time (T*)	Selling price(p*)
+20%	1001.98	0.298689	183.864
+15%	1001.98	0.298691	183.864
+10%	1001.98	0.298691	183.863
+5%	1001.98	0.298692	183.863
0%	1001.97	0.298689	183.864
-5%	1001.989	0.298691	183.862
-10%	1001.99	0.298691	183.862
-15%	1002	0.298691	183.862
-20%	1002.1	0.298689	18.861

Table 3
Sensitivity analysis for carbon related holding rate (h_e)

h_e	Total Cost ($TC_{1.1}^{*}$)	Cycle Time (T*)	Selling price(p*)
+20%	1001.99	0.298685	183.863
+15%	1001.99	0.298687	183.863
+10%	1001.99	0.298688	183.863
+5%	1001.98	0.298688	183.863
0%	1001.97	0.298689	183.864
-5%	1001.97	0.298689	183.864
-10%	1001.97	0.298696	183.864
-15%	1001.97	0.298696	183.864
-20%	1001.97	0.298694	183.864

Table 4
Sensitivity analysis for carbon related ordering rate (A_e)

A_e	Total Cost ($TC_{1.1}^{*}$)	Cycle Time (T*)	Selling price(p*)
+20%	1001.99	0.298728	183.863
+15%	1001.99	0.29872	183.863
+10%	1001.99	0.298712	183.863
+5%	1001.98	0.298701	183.863
0%	1001.97	0.298689	183.864
-5%	1001.97	0.298681	183.864
-10%	1001.97	0.298672	183.864
-15%	1001.97	0.298663	183.864
-20%	1001.97	0.298654	183.864

Table 5
Sensitivity analysis for carbon tax rate (λ).

λ	Total Cost ($TC_{1.1}^{*}$)	Cycle Time (T*)	Selling price(p*)
+20%	1001.97	0.298725	183.864
+15%	1001.97	0.298717	183.864
+10%	1001.97	0.298706	183.864
+5%	1001.97	0.298696	183.864
0%	1001.97	0.298689	183.864
-5%	1001.98	0.298681	183.865

-10%	1001.99	0.298674	183.865
-15%	1002	0.298667	183.865
-20%	1002.1	0.298656	183.865

Table 6
Sensitivity analysis for variations in inflation rate (r)

R	Total Cost ($TC^*_{1.1}$)	Cycle Time (T*)	Selling price(p*)
+20%	1001.81	0.270465	183.862
+15%	1001.82	0.276405	183.862
+10%	1001.84	0.283271	183.863
+5%	1001.91	0.290677	183.863
0%	1001.97	0.298689	183.864
-5%	1002.06	0.307402	183.864
-10%	1002.14	0.316919	183.864
-15%	1002.22	0.327364	183.864
-20%	1002.31	0.338922	183.865

Table 7
Sensitivity analysis for variations in permissible delay period(M)

M	Total Cost ($TC^*_{1.1}$)	Cycle Time (T*)	Selling price(p*)
+20%	1002.31	0.263753	183.861
+15%	1002.22	0.273526	183.861
+10%	1002.13	0.282563	183.862
+5%	1002.06	0.290932	183.863
0%	1001.97	0.298689	183.864
-5%	1001.92	0.305886	183.864
-10%	1001.86	0.312556	183.864
-15%	1001.75	0.318736	183.864
-20%	1001.31	0.324454	183.865

5.2 Sensitivity analysis for shortage w.r.t. various parameter

Table 8
Sensitivity analysis for variations in ordering cost (A)

A	Total Cost ($TC^*_{2.2}$)	Time (T*)	Time(t_1^*)	Selling price(p)
+20%	1065.73	0.158932	0.136732	104.435
+15%	1065.87	0.155306	0.132476	104.791
+10%	1065.91	0.154287	0.131236	104.898
+5%	1065.92	0.149856	0.127845	105.238
0%	1065.99	0.140191	0.123907	105.772
-5%	1066.31	0.139872	0.121534	105.869
-10%	1089.32	0.138648	0.121365	105.977
-15%	1158.2	0.137534	0.121289	106.057
-20%	1141.13	0.137432	0.121187	106.069

Table 9
Sensitivity analysis for carbon emission deterioration rate (C_e)

C_e	Total Cost ($TC^*_{2.2}$)	Time (T*)	Time(t_1^*)	Selling price(p*)
+20%	1064.51	0.140174	0.123686	105.79
+15%	1064.7	0.140181	0.123724	105.786
+10%	1064.88	0.140185	0.123798	105.779
+5%	1065.07	0.140188	0.123871	105.772
0%	1065.99	0.140191	0.123907	105.772
-5%	1066.14	0.140211	0.123912	105.768
-10%	1066.44	0.140227	0.123967	105.761
-15%	1066.81	0.140265	0.123989	105.759
-20%	1066.99	0.140298	0.123995	105.758

Table 10
Sensitivity analysis for carbon related holding rate (h_e)

h_e	Total Cost ($TC^*_{2.2}$)	Time (T*)	Time(t_1^*)	Selling price(p)
+20%	1064.49	0.140191	0.123907	105.772
+15%	1064.58	0.140191	0.123907	105.772
+10%	1064.99	0.140191	0.123907	105.772
+5%	1065.27	0.140191	0.123907	105.772
0%	1065.99	0.140191	0.123907	105.772

-5%	1066.23	0.140191	0.123907	105.772
-10%	1066.63	0.140191	0.123907	105.772
-15%	1066.78	0.140191	0.123907	105.772
-20%	1066.99	0.140191	0.123907	105.772

Table 11
Sensitivity analysis for carbon related ordering rate (A_e)

A_e	Total Cost ($TC_{2.2}^*$)	Time (T*)	Time(t_1^*)	Selling price(p*)
+20%	1065.99	0.140191	0.123907	105.772
+15%	1065.99	0.140191	0.123907	105.772
+10%	1065.99	0.140191	0.123907	105.772
+5%	1065.99	0.140191	0.123907	105.772
0%	1065.99	0.140191	0.123907	105.772
-5%	1065.99	0.140191	0.123907	105.772
-10%	1065.99	0.140191	0.123907	105.772
-15%	1065.99	0.140191	0.123907	105.772
-20%	1065.99	0.140191	0.123907	105.772

Table 12
Sensitivity analysis for carbon tax rate (λ)

λ	Total Cost ($TC_{2.2}^*$)	Time (T*)	Time(t_1^*)	Selling price(p*)
+20%	1065.07	0.140174	0.123724	105.7873
+15%	1065.63	0.140175	0.123734	105.785
+10%	1065.44	0.140181	0.123798	105.78
+5%	1065.81	0.140188	0.123871	105.773
0%	1065.99	0.140191	0.123907	105.772
-5%	1066.12	0.140193	0.123913	105.771
-10%	1066.37	0.140194	0.123915	105.768
-15%	1066.49	0.140196	0.123916	105.767
-20%	1066.74	0.140198	0.123919	105.766

Table 13
Sensitivity analysis for variations in inflation rate (r)

R	Total Cost ($TC_{2.2}^*$)	Time (T*)	Time(t_1^*)	Selling price(p*)
+20%	1065.01	0.138641	0.122356	106.717
+15%	1065.42	0.139842	0.123558	106.401
+10%	1065.56	0.140021	0.123737	106.195
+5%	1065.67	0.140181	0.123897	105.98
0%	1065.99	0.140191	0.123907	105.772
-5%	1066.23	0.142853	0.125569	105.353
-10%	1066.38	0.143191	0.126907	105.01
-15%	1066.63	0.144346	0.128062	104.655
-20%	1066.75	0.145324	0.129180	104.295

6. Result and discussion

1. From Table 1 & 8 we observe that ordering cost sensitive parameters decrease then average profit increase, optimal cycle length increases and optimal selling price remains almost unchanged.
2. From Table 2 & 9 we observe that carbon emission deterioration rate sensitive parameters decrease, then average profit increases, optimal cycle length and optimal selling price remains almost unchanged.
3. From Table 2 & 10 we observe that carbon emission related holding rate sensitive parameters decrease then average profit increase, optimal cycle length and optimal selling price remains almost unchanged.
4. From Table 3 & 11 we observe that carbon emission related ordering rate sensitive parameters decrease then average profit, optimal cycle length and optimal selling price remains almost unchanged. There is no effect of carbon ordering sensitive parameters in profit.
5. From Table 4 & 12 we observe that carbon tax sensitive parameters decrease then average profit, optimal cycle length increases and optimal selling price remains almost unchanged.
6. From Table 5 & 13 we observe that inflation rate sensitive parameters decrease then average profit, optimal cycle length increases and optimal selling price slightly change.
7. From Table 6 & 14 we observe that trade credit sensitive parameters decrease then average profit, optimal cycle length increases and optimal selling price slightly change.

7. Conclusion

Increasing profit as well as protecting the environment and using natural resources efficiently are a few reasons why companies seek to implement a proper inventory plan for perishable items. This study we observe that carbon tax policy is an important role play to control the carbon emission but there total average cost increase whenever increase carbon tax rate. It analyzes two sustainable inventory procedures when adopting a trade credit policy for both time-sensitive and delayed decay items. There is no stock out situation in the first inventory procedure, while a stock out situation exists in the second inventory procedure an associated backlogging rate is adopted based on waiting time. Among the demand patterns considered in the current study, only linear increases or decreases are considered based on the current selling price. In both inventory procedures, the practitioner's primary goal is to maximize profits. Moreover, sensitivity analyses and numerical findings are used to make some insights. At the end of an inventory cycle, the industry manager can sell some items with a salvage value after storing the highest number of products in the warehouse. Lastly, the decision maker may not always be able to afford to sell the products at the regular price. A company manager can store the most amount of goods possible in a warehouse and sell some of those goods with salvage value at the end of the cycle if there is a decrease in the varying cost coefficient in the cost per unit of holding. Further exploration could be done by converting all inventory costs to interval values or fuzzy values.

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