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Equilibrium solution of non-cooperative multiobjective bimatrix game of Z-numbers

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ABSTRACT

In this paper we introduce the concept of equilibrium for a non-cooperative multiobjective bimatrix game with payoff matrices and goals of Z-Numbers. In the recent studies of the authors, the problem of finding equilibrium for a non-cooperative bimatrix of Z-Numbers are investigated. Multiple payoffs are often dealt with in games because a decision making problem under conflict usually involves multiple objectives or attributes such as cost, time and productivity. We let each of the objectives of the problem correspond to each of the payoffs of the game. To aggregate multiple goals, we employ two basic methods, one by weighting coefficients and the other by a minimum component. In order to find the equilibrium solution in such circumstances, we developed a mathematical programming problem to maximize the aggregated goal subject to constraint of satisfying an aspiration level of confidence in the equilibrium solution. Finally a method is presented to determine the equilibrium solution with respect to the level of achievement to the aggregated goal.

1. Introduction

This study develops fuzzy non-cooperative game theory by considering two important concepts. First, the fact that in real-world circumstances, each player has to consider several objectives in making decision on various strategies and second, lack of full confidence or trust in the fuzzy expressions. Campos (1989) conducted the first study in fuzzy payoffs matrix game. In his study, the problem of finding an equilibrium solution to the zero-sum fuzzy matrix game was changed to solving a pair of fuzzy linear programming problem using Yager's method (Yager, 1981). Then Campos and Gonzalez (1999) developed a model in the state of using non-linear ranking function (Campos et al., 1992). Presenting a definition of feasible solutions and the fuzzy optimal solution, Li (1999); Li and Yang (2003) and Bector et al. (2004) developed multi-objective linear programming models and suggested a two-level programming method to solve the problems. In the model developed by Vijay et al. (2004), constraints required for feasible solutions are first defined, then considering the ranking function, a fuzzy non-linear programming model is developed and like the Bector's model, this model is also changed to a crisp non-

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linear programming model. Then Vijay et al. (2005, 2007) developed their model to solve two-person finite games using fuzzy goals and payoffs. Maeda (2003) suggested another method by changing the problem into a crisp parametric game by considering payoffs as symmetric triangular fuzzy numbers, dividing fuzzy numbers into two sections, crisp and fuzzy, and considering the concept of α-cuts. Identifying the Nash equilibrium of this problem using certain values of parameters, the dominant minimax equilibrium strategy was achieved. Cunlin and Qiang (2011) developed Maeda's model where payoffs were asymmetric triangular fuzzy numbers. Lui and Kao (2007) developed a method for zero-sum games based on the extension principle and α-cuts. In addition, Buckley and Jowers (2008) suggested a model similar to Lui and Kao's model. In their model, the optimal strategy was obtained using Monte Carlo simulation where there was no saddle point. In the model of Xu and Zhao (2005), payoffs were fuzzy variables. In their first study, they defined the measures of possibility, credibility and fuzzy expected value and then, three types of minimax equilibrium strategies, r-possible, r-credible, and expected were introduced. These equilibrium strategies were obtained using an iterative algorithm based on fuzzy simulation. Xu and Zhao (2006) considered payoffs as random fuzzy variables. This means that all payoffs have a membership function, parameters of which are randomly characterized by the distribution function. Then they defined an algorithm to estimate the fuzzy expected value and to identify optimal strategies. In another study investigating a two-person zero-sum game with payoffs in the form of fuzzy variables, Xu and Wang (2009) defined the pessimistic and optimistic values of minimax equilibrium strategy in the confidence level of α, modeled it using fuzzy linear programming model and obtained the optimal equilibrium strategy using particle swarm optimization (PSO) algorithm and fuzzy simulation. In another study, Xu and Li (2010) considered payoffs as fuzzy random variables. This means that each payoff is a random variable, the probability density function parameters of which are fuzzy. In addition, they defined an algorithm to estimate the fuzzy expected value and to identify optimal strategies. In the approach of Gao-Sheng et al. (2011), each player's expected payoff, in a state that payoffs are considered as triangular fuzzy numbers, is calculated and explicitly solved using the definition of fuzzy expected value defined based on credibility measures.

Nishizaki and Sakawa (2000) studied multi-objective bimatrix games with fuzzy goals and payoffs. In their approach, the level of achievement to each objective was defined based on goals for each mixed strategy, and a model based on nonlinear programming was suggested to achieve the highest degree of achieving the aggregated goal, resulting from aggregating different goals of the problem, as the objective function of the problem. In mentioned studies, uncertainty in strategies and payoffs was investigated using the concepts of decision making in fuzzy environment or fuzzy mathematics. However, the effect of the level of confidence or trust in the expression of numbers or fuzzy sets was not considered. In fact, two levels of uncertainty were discussed in this study. As in previous studies, the first level considers fuzzy numbers and sets in the definition of payoffs, and the level of trust in these ambiguous information is expressed in the upper level. These two levels are investigated with the concept of Z-number defined by Zadeh (2011) for the first time. Akhbari and Sadinejad (2015) developed their game in bimatrix games, by considering payoffs as Z-numbers and modeled the problem using Nishizaki and Sakawa’s method, and the proposed study of this paper develops their work to find equilibrium strategy of multi-objective non-cooperative games.

The paper is organized as follows, In Section 2, we describe some basic concepts of multi-objective non-cooperative bimatrix game and Z-Numbers and introduce some basic definitions and notations on bimatrix games with payoffs and goals of z-numbers. In Section 3, we focus on developing an optimization model that gives the Equilibrium Solution for this game. In Section 4, two examples are given to illustrate the solution procedure developed here for solving such games.

2. Non-cooperative game with payoffs and goals of Z-Numbers

In this section, we first consider the concept of the z-number and then investigate a non-cooperative moltiobjective bimatrix game with payoffs of z-numbers. A z-number, Z = (A, B), is an order pair of two fuzzy numbers. The number, A, is a restriction on the values which the real-value variable, X, can allocate to itself. The second number, B, is a restriction on degree of trust that X is A. A and B are usually described in natural language e.g. (about 45 dollars, very sure) (Zadeh, 2011). Let $\mathrm{I} = \{1, \dots, \mathrm{m}\}$ and $\mathrm{J} = \{1, \dots, \mathrm{n}\}$ sets are pure strategies of players I and II, respectively. Mixed strategies of these two players showing the probability distributions of all pure strategies are defined as follows:

$$\mathrm{x} = (\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{m})^\mathrm{T} \in \mathrm{X} \triangleq \{\mathrm{x} \in \mathrm{R}_+^\mathrm{m} | \textstyle\sum_{\mathrm{i}\in\mathrm{I}} \mathrm{x}_\mathrm{i} = 1\}, \tag{1}$$

$$\mathrm{y} = (\mathrm{y}_1, \dots, \mathrm{y}_\mathrm{n})^\mathrm{T} \in \mathrm{Y} \triangleq \{\mathrm{y} \in \mathrm{R}_+^\mathrm{n} | \textstyle\sum_{\mathrm{j}\in\mathrm{J}} \mathrm{y}_\mathrm{i} = 1\} \tag{2}$$

where x and y are mixed strategies of players I and II, respectively and $\mathrm{R}_+^\mathrm{m} = \{\mathrm{r} \in \mathrm{R}^\mathrm{m} | \mathrm{r}_\mathrm{i} \geq 0, \mathrm{i} = 1, \dots, \mathrm{m}\}$. Let an index set of all the objectives of Player I be $\triangleq \{1, \dots, s\}$ and that of Player II be $L \triangleq \{1, \dots, r\}$, When Player I chooses a pure strategy $i \in I$ and Player II chooses a pure strategy $j \in J$ multiple payoffs of Players I and II are $\left((\tilde{a}_{ij}^1, \tilde{c}_{ij}^1), \dots, (\tilde{a}_{ij}^s, \tilde{c}_{ij}^s)\right)$and $\left((\tilde{b}_{ij}^1, \tilde{e}_{ij}^1), \dots, (\tilde{b}_{ij}^r, \tilde{e}_{ij}^r)\right)$, respectively. Then a multiobjective bimatrix game with payoffs of z-numbers is represented as payoff matrices:

$$A = \begin{bmatrix} ((\tilde{a}_{11}^1, \tilde{c}_{11}^1), \dots, (\tilde{a}_{11}^s, \tilde{c}_{11}^s)) & \cdots & ((\tilde{a}_{1n}^1, \tilde{c}_{1n}^1), \dots, (\tilde{a}_{1n}^s, \tilde{c}_{1n}^s)) \\ \vdots & \ddots & \vdots \\ ((\tilde{a}_{m1}^1, \tilde{c}_{m1}^1), \dots, (\tilde{a}_{m1}^s, \tilde{c}_{m1}^s)) & \cdots & ((\tilde{a}_{mn}^1, \tilde{c}_{mn}^1), \dots, (\tilde{a}_{mn}^s, \tilde{c}_{mn}^s)) \end{bmatrix}, \tag{3}$$

$$B = \begin{bmatrix} ((\tilde{b}_{11}^1, \tilde{e}_{11}^1), \dots, (\tilde{b}_{11}^r, \tilde{e}_{11}^r)) & \cdots & ((\tilde{b}_{1n}^1, \tilde{e}_{1n}^1), \dots, (\tilde{b}_{1n}^r, \tilde{e}_{1n}^r)) \\ \vdots & \ddots & \vdots \\ ((\tilde{b}_{m1}^1, \tilde{e}_{m1}^1), \dots, (\tilde{b}_{m1}^r, \tilde{e}_{m1}^r)) & \cdots & ((\tilde{b}_{mn}^1, \tilde{e}_{mn}^1), \dots, (\tilde{b}_{mn}^r, \tilde{e}_{mn}^r)) \end{bmatrix}. \tag{4}$$

or

$$\begin{aligned} A^k &= \begin{bmatrix} (\tilde{a}_{11}^k, \tilde{c}_{11}^k) & \cdots & (\tilde{a}_{1n}^k, \tilde{c}_{1n}^k) \\ \vdots & \ddots & \vdots \\ (\tilde{a}_{m1}^k, \tilde{c}_{m1}^k) & \cdots & (\tilde{a}_{mn}^k, \tilde{c}_{mn}^k) \end{bmatrix}, k \in K, \\ B^l &= \begin{bmatrix} (\tilde{b}_{11}^l, \tilde{e}_{11}^l) & \cdots & (\tilde{b}_{1n}^l, \tilde{e}_{1n}^l) \\ \vdots & \ddots & \vdots \\ (\tilde{b}_{m1}^l, \tilde{e}_{m1}^l) & \cdots & (\tilde{b}_{mn}^l, \tilde{e}_{mn}^l) \end{bmatrix}, l \in L. \end{aligned} \tag{5}$$

Let $A = (A^1, \dots, A^s)$ and $B = (B^1, \dots, B^r)$, and then the multiobjective bimatrix game with payoffs of z-number is defined by *(A,B)*. It is assumed that the fuzzy numbers of $\tilde{a}_{ij}^k$, $\tilde{b}_{ij}^l$, $\tilde{c}_{ij}^k$ and $\tilde{e}_{ij}^l$ are of LR-type fuzzy numbers and are represented as follows:

$$\tilde{a}_{ij}^k = (a_{ij}^k, \alpha_{ij}^k, \beta_{ij}^k), \alpha_{ij}^k, \beta_{ij}^k \geq 0, k \in K,$$
$$\tilde{b}_{ij}^l = (b_{ij}^l, \nu_{ij}^l, \omega_{ij}^l), \nu_{ij}^l, \omega_{ij}^l \geq 0, l \in L,$$
$$\tilde{c}_{ij}^k = (c_{ij}^k, \gamma_{ij}^k, \delta_{ij}^k), 0 \leq c_{ij}^k, \gamma_{ij}^k, \delta_{ij}^k \leq 1, 0 \leq c_{ij}^k - \gamma_{ij}^k \leq 1, 0 \leq c_{ij}^k + \delta_{ij}^k \leq 1, k \in K,$$
$$\tilde{e}_{ij}^l = (e_{ij}^l, \varepsilon_{ij}^l, \sigma_{ij}^l), 0 \leq e_{ij}^l, \varepsilon_{ij}^l, \sigma_{ij}^l \leq 1, 0 \leq e_{ij}^l - \varepsilon_{ij}^l \leq 1, 0 \leq e_{ij}^l + \sigma_{ij}^l \leq 1, l \in L.$$

And their membership functions are as follows:

$$\mu_{\tilde{a}_{ij}^k}(p) = \begin{cases} 0 \quad if\ p < a_{ij}^k - \alpha_{ij}^k \\ (p - a_{ij}^k + \alpha_{ij}^k)/\alpha_{ij}^k \ \ if\ a_{ij}^k - \alpha_{ij}^k \leq p < a_{ij}^k \\ (a_{ij}^k + \beta_{ij}^k - p)/\beta_{ij}^k \ \ if\ a_{ij}^k \leq p \leq a_{ij}^k + \beta_{ij}^k \\ 0 \quad if\ a_{ij}^k + \beta_{ij}^k < p \end{cases} \tag{6}$$

$$\mu_{\tilde{b}_{ij}^l}(p) = \begin{cases} 0 \quad if\ p < b_{ij}^l - \nu_{ij}^l \\ (p - b_{ij}^l + \nu_{ij}^l)/\nu_{ij}^l \ \ if\ b_{ij}^l - \nu_{ij}^l \leq p < b_{ij}^l \\ (b_{ij}^l + \omega_{ij}^l - p)/\omega_{ij}^l \ \ if\ b_{ij}^l \leq p \leq b_{ij}^l + \omega_{ij}^l \\ 0 \quad if\ b_{ij}^l + \omega_{ij}^l < p \end{cases} \tag{7}$$

$$\mu_{c_{ij}^k}(p) = \begin{cases} 0 \quad if\ p < c_{ij}^k - \gamma_{ij}^k \\ (p - c_{ij}^k + \gamma_{ij}^k)/\gamma_{ij}^k \ \ if\ c_{ij}^k - \gamma_{ij}^k \le p < c_{ij}^k \\ (c_{ij}^k + \delta_{ij}^k - p)/\delta_{ij}^k \ \ if\ c_{ij}^k \le p \le c_{ij}^k + \delta_{ij}^k \\ 0 \quad if\ c_{ij}^k + \delta_{ij}^k < p \end{cases} \tag{8}$$

$$\mu_{e_{ij}^l}(p) = \begin{cases} 0 \quad if\ p < e_{ij}^l - \varepsilon_{ij}^l \\ (p - e_{ij}^l + \varepsilon_{ij}^l)/\varepsilon_{ij}^l \ \ if\ e_{ij}^l - \varepsilon_{ij}^l \le p < e_{ij}^l \\ (e_{ij}^l + \sigma_{ij}^l - p)/\sigma_{ij}^l \ \ if\ e_{ij}^l \le p \le e_{ij}^l + \sigma_{ij}^l \\ 0 \quad if\ e_{ij}^l + \sigma_{ij}^l < p \end{cases} \tag{9}$$

The expected value of the game for player I can be expressed as $(\tilde{E}_1(x,y), \widetilde{Cr}_1(x,y))$ in which the first and the second components show the vector of fuzzy expected value of the game and the level of confidence, respectively. The values of these two elements for player I are obtained as follows:

$$\tilde{E}_1(x,y) = (\tilde{E}_1^1(x,y), \tilde{E}_1^2(x,y), \dots, \tilde{E}_1^s(x,y)) \tag{10}$$

In which, for the mixed strategies *(x,y)*, the kth fuzzy expected payoff for Player I is denoted by $\tilde{E}_1^k(x,y), k = 1,2, \dots, s, and\ can\ be\ optained\ by$:

$$\tilde{E}_1^k(x,y) = x^T \tilde{A}^k y,\ k \in K, \tag{11}$$

where $\tilde{A}^k$ is,

$$\tilde{A}^k = \begin{bmatrix} \tilde{a}_{11}^k & \dots & \tilde{a}_{1n}^k \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1}^k & \dots & \tilde{a}_{mn}^k \end{bmatrix}, k \in K. \tag{12}$$

and

$$\widetilde{Cr}_1(x,y) = x^T \tilde{C} y = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_i y_j \tag{13}$$

where $\tilde{c}_{ij}$ is obtained by a *t-norm* operator (here *min):*

$$\tilde{c}_{ij} = \prod_{1}^{k} \tilde{c}_{ij}^k = \min_{k \in K} \tilde{c}_{ij}^k. \tag{14}$$

Using the fuzzy extension principle, fuzzy membership functions for the elements of the vector of fuzzy expected value for player I can be obtained as follows:

$$\mu_{\tilde{E}_1^k(x,y)}(p) = \sup_{p = x^T \tilde{A}^k y} \min_{i,j} \mu_{\tilde{a}_{ij}^k}(p), k \in K. \tag{15}$$

Accordingly, the fuzzy expected value for player I is represented as a L_R fuzzy number as follows:

$$\tilde{E}_1^k(x,y) = \left(x^T A^k y, x^T \acute{A}^k y, x^T \grave{A}^k y\right), k \in K. \tag{16}$$

where $\acute{A}^k$ and $\grave{A}^k$ are $m \times n\ matrices$, the elements of which are α_{ij}^k and β_{ij}^k, respectively, and their membership function is as follows:

$$\mu_{\tilde{E}_1^k(x,y)}(p) = \begin{cases} 0 \quad if\ p < x^T(A^k - \acute{A}^k)y \\ (p - x^T(A^k - \acute{A}^k)y)/x^T \acute{A}^k y \ \ if\ x^T(A^k - \acute{A}^k)y \le p < x^T A^k y \\ (x^T(A^k + \grave{A}^k)y - p)/x^T \grave{A}^k y \ \ if\ x^T \grave{A}^k y \le p \le x^T(A^k + \grave{A}^k)y \\ 0 \quad if\ x^T(A^k + \grave{A}^k)y < p \end{cases} \tag{17}$$

or,

$$\mu_{\tilde{E}_1^k(x,y)}(p)=\begin{cases} 0 \quad if\ p<\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k-\alpha_{ij}^k)x_i\,y_j \\ (p-\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k-\alpha_{ij}^k)x_i\,y_j)/\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}^k x_i\,y_j \ if\ \sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k-\alpha_{ij}^k)x_i\,y_j\le p<\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^k x_i\,y_j \\ (\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k+\beta_{ij}^k)x_i\,y_j-p)/\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}^k x_i\,y_j \ if\ \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^k x_i\,y_j\le p\le\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k+\beta_{ij}^k)x_i\,y_j \\ 0 \quad if\ \sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k+\beta_{ij}^k)x_i\,y_j<p \end{cases} \tag{18}$$

The expected value of the game for player II, $\left(\tilde{E}_2(x,y),\widetilde{Cr}_2(x,y)\right)$can be similarly obtained.

3. An equilibrium solution in multiobjective games game with Payoffs and goals of Z-Numbers

In the present study, since the expected value of each player's is of z-numbers type, we consider a goal of z-numbers type for them. Assume that for player I, we consider a goal of z-number,$\left(\tilde{G}_1,\tilde{R}_1\right)$, in which $\tilde{G}_1$ is the vector of fuzzy numbers indicating the desired payoffs of the game :

$$\tilde{G}_1=(\tilde{G}_1^1,\tilde{G}_1^2,\dots,\tilde{G}_1^s). \tag{19}$$

For the *k*th payoff, let Player I's fuzzy goal $\tilde{G}_1^k$, , $k\in K$ be a fuzzy set on the set R characterized by the membership function $\mu_{\tilde{G}_1^k}$:

$$\mu_{\tilde{G}_1^k}(p)=\begin{cases} 0 & p\le\underline{a}^k \\ \dfrac{p-\underline{a}^k}{\overline{a}^k-\underline{a}^k} & \underline{a}^k\le p\le\overline{a}^k, k\in K. \\ 1 & p\ge\overline{a}^k \end{cases} \tag{20}$$

Accordingly, if the payoff of player I is less than $\underline{a}^k$, its membership degree in $\tilde{G}_1^k$will be equal to zero and not desirable and will be linearly increased in the distance between $\underline{a}^k$ and $\overline{a}^k$, and the membership degree or the maximum desirability is obtained in amounts higher than maximum $\overline{a}^k$. Therefore, membership degree in $\tilde{G}_1^k$ indicates the level of achievement to the goal.

Since we do not want to limit the improvement of players' payoffs, we identify the values of parameters $\underline{a}$ and $\overline{a}$ so that the following equation is always true:

$$0<\frac{x^T(A^k+\grave{A}^k)y-\underline{a}^k}{\overline{a}^k-\underline{a}^k+x^T\grave{A}^k y}<1 \tag{21}$$

Therefore, we can define the values of parameters $\underline{a}^k$ and $\overline{a}^k$ as follows:

$$\underline{a}^k=\min_{i,j}\{a_{ij}^k-\alpha_{ij}^k\} \tag{22}$$

$$\overline{a}^k=\max_{i,j}\{a_{ij}^k+\beta_{ij}^k\} \tag{23}$$

In fact, the second part of the goal or $\tilde{R}_1$ shows the acceptable level of trust or confidence of player I in the solution which means player I accepts only solution that the level of the confidence of it, $\widetilde{Cr}_1$, is greater than $\tilde{R}_1$. $\tilde{R}_1$ is a fuzzy set defined as a L-R fuzzy number on the set of real numbers in interval [0,1]. Since confidence in the solution is resulted from fuzzy numbers of payoff confidence, at least one of these values or $\tilde{c}_{ij}$s should be higher than or equal $\tilde{R}_1$, or in other words $\tilde{R}_1$should be smaller than the maximum $\tilde{c}_{ij}$s to have at least one feasible solution. The set of possible fuzzy values of $\tilde{R}_1$ is called H_1. Here, the greater relationship is defined through the comparison of parties' desuzzified values calculated by center of gravity method.

$$\tilde{R}_1 = (R_1, l_1, r_1) \in H_1 = \{\tilde{h} = (h, l, r) | \tilde{h} \leq max(\tilde{c}_{ij}), h - l \geq 0, h + r \leq 1\} \tag{24}$$

Membership function $\tilde{R}_1$ is defined as follows:

$$\mu_{\tilde{R}_1}(p) = \begin{cases} 0 & if \quad p < R_1 - l_1 \\ (p - R_1 + l_1)/l_1 & if \quad R_1 - l \leq p < R_1 \\ (R_1 + r_1 - p)/r_1 & if \quad R_1 \leq p \leq R_1 + r_1 \\ 0 & if \quad R_1 + r_1 < p \end{cases} \tag{25}$$

The minimum level of trust or confidence of player II ($\tilde{R}_2 = (R_2, l_2, r_2)$) can be similarly obtained. Accordingly, the set of feasible solutions (S) can be defined as follows:

$$S = \{(x, y) | x \in X, y \in Y, \widetilde{Cr}_1(x, y) \gtreqqless \tilde{R}_1, \widetilde{Cr}_2(x, y) \gtreqqless \tilde{R}_2\} \tag{26}$$

Considering the method of ranking numbers using the center of gravity, $\widetilde{Cr}_1(x, y) \gtreqqless \tilde{R}_1$ can be modified as follows:

$$(3R_1 - l_1 + r_1) - \sum_{i=1}^{m} \sum_{j=1}^{n} (3c_{ij} - \delta_{ij} + \gamma_{ij}) x_i y_j^* \leq 0 \rightarrow$$

$$1 - \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{3c_{ij} - \delta_{ij} + \gamma_{ij}}{3R_1 - l_1 + r_1} x_i y_j^* \leq 0 \rightarrow 1 - x^T K y^* \leq 0$$

Where K are $m \times n \; matrix$, the elements of which, k_{ij} is obtained as follows:

$$k_{ij} = \frac{3c_{ij} - \delta_{ij} + \gamma_{ij}}{3R_1 - l_1 + r_1}.$$

Similarly, in $\widetilde{Cr}_2(x, y) \gtreqqless \tilde{R}_2$, we have $1 - x^{*T} M y \leq 0$. According to the definition of the set of feasible solutions (S), the level of achievement to the first player's *k*th goal or $d_1^k(x, y)$ can be defined as follows:

$$d_1^k(x, y) = \max_p min\{\mu_{\tilde{E}_1^k(x,y)}(p), \mu_{\tilde{G}_1^k}(p)\}, (x, y) \in S. \tag{27}$$

or

$$d_1^k(x, y) = \begin{cases} \dfrac{x^T (A^k + \grave{A}^k) y - \underline{a}^k}{\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y} & if \; (x, y) \in S \\ 0 & if \; (x, y) \notin S \end{cases} \tag{28}$$

The level of achievement to the second player's *l*th goal or $d_2^l(x, y)$ can be similarly defined.
To aggregate multiple fuzzy goals, we employ two basic methods, one by weighting coefficients and the other by a minimum component.

3.1. Aggregation by weighting coefficients

Let Player *I*'s and *II*'s weighting coefficients for fuzzy goals be $\vartheta \in \{\vartheta \in R_+^r | \sum_{k=1}^r \vartheta_k = 1\}$ and $\omega \in \{\omega \in R_+^s | \sum_{k=1}^s \omega_k = 1\}$, respectively. Then Player I's aggregated fuzzy goals is represented by

$$Wd_1(x, y) = \sum_{k=1}^{r} \vartheta_k d_1^k(x, y) \tag{29}$$

The degree of attainment of the lth fuzzy goal for Player II can be defined in a similar way. We now consider equilibrium solutions with respect to the degree of attainment of the aggregated fuzzy goal. A pair of strategies x^* and y^*,$(x^*, y^*) \in S,$ is said to be an equilibrium solution if for any other mixed strategies x and y, $(x, y) \in S$,

$$\sum_{k=1}^{r} \vartheta_k d_1^k(x^*, y^*) \geq \sum_{k=1}^{r} \vartheta_k d_1^k(x, y^*),$$
$$\sum_{l=1}^{s} \omega_l d_2^l(x^*, y^*) \geq \sum_{l=1}^{s} \omega_l d_2^l(x^*, y). \tag{30}$$

We will examine a relation between equilibrium solutions and optimal solutions to a certain mathematical programming problem. Employing aggregation by weighting coefficients, for a pair of x and y, Player I's degree of attainment of the aggregated fuzzy goal can be represented by

$$Wd_1(x, y) = \sum_{k=1}^{r} \vartheta_k \frac{x^T(A^k + \grave{A}^k)y^* - \underline{a}^k}{\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y^*} = \sum_{k=1}^{r} \vartheta_k \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij}^k + \grave{a}_{ij}^k) x_i y_j - \underline{a}^k}{\overline{a}^k - \underline{a}^k + \sum_{i=1}^{m} \sum_{j=1}^{n} \grave{a}_{ij}^k x_i y_j}$$

and Player II's can also be represented similarly. A pair of optimal solutions $x^*$and $y^*$to the following two mathematical programming problems is an equilibrium solution.

$$Wd_1(x^*, y^*) = \max_x Wd_1(x, y^*), \tag{31}$$
subject to
$$e^{mT}x - 1 = 0,$$
$$1 - x^T K y^* \leq 0,$$
$$x \geq 0^m,$$
$$Wd_2(x^*, y^*) = \max_x Wd_1(x^*, y), \tag{32}$$
subject to
$$e^{nT}y - 1 = 0,$$
$$1 - x^{*T} M y \leq 0,$$
$$y \geq 0^n.$$

Accordingly, the two problems can be rewritten as follows:

$$\max \sum_{k=1}^{r} \vartheta_k \frac{x^T(A^k + \grave{A}^k)y^* - \underline{a}^k}{\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y^*}$$
subject to
$$1 - x^T K y^* \leq 0,$$
$$e^{mT}x = 1,$$
$$x \geq 0^m, \tag{33}$$
and
$$\max \sum_{l=1}^{s} \omega_l \frac{x^{*T}(B^l + \grave{B}^l)y - \underline{b}^l}{\overline{b}^l - \underline{b}^l + x^{*T} \grave{B}^l y}$$
subject to
$$1 - x^{*T} M y \leq 0,$$
$$e^{mT}y = 1,$$
$$y \geq 0^n. \tag{34}$$

By applying the necessary Kuhn-Tucker Conditions for the problems (33) and (34), a necessary condition that a pair of x and y be an equilibrium solution with respect to the degree of attainment of the fuzzy goal becomes that there exist scalar values λ and $\acute{\lambda}$ and positive values τ and $\acute{\tau}$ such that x, y, $\lambda, \acute{\lambda}, \tau$ *and* $\acute{\tau}$ satisfy

$$\sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^T (A^k + \grave{A}^k) y - \underline{a}^k x^T A^k y}{\left(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y\right)^2} + \lambda + \tau = 0, \tag{35}$$

$$\sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^T (B^l + \grave{B}^l) y - \underline{b}^l x^T B^l y\right)}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda} + \acute{\tau} = 0, \tag{36}$$

$$\sum_{k=1}^{r} \vartheta_k \frac{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y) A^k y + (\overline{a}^k - x^T A^k y) \grave{A}^k y}{\left(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y\right)^2} + \lambda e^m + \tau K y \leq 0, \tag{37}$$

$$\sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right) B^l y + \left(\overline{b}^l - x^T B^l y\right) \grave{B}^l y}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda} e^n + \acute{\tau} x^T M \leq 0, \tag{38}$$

$$1 - x^T K y \leq 0, \tag{39}$$

$$1 - x^T M y \leq 0 \tag{40}$$

$$e^{mT} x - 1 = 0, \tag{41}$$

$$e^{nT} y - 1 = 0, \tag{42}$$

$$\tau (1 - x^T K y) = 0, \tag{43}$$

$$\acute{\tau} (1 - x^T M y) = 0, \tag{44}$$

$$x \geq 0^m, \tag{45}$$

$$y \geq 0^n. \tag{46}$$

where $\grave{A}^k$ is an $m \times n$ matrix the *ij*-element of which is $\grave{a}_{ij}^k$, $\grave{B}^l$ is a similar matrix , λ *and* $\acute{\lambda}$ are scalar variables and τ *and* $\acute{\tau}$ are positive variables.

Lemma 1. x and y satisfy the Kuhn-Tucker Conditions (35)-(46) if and only if there exists an optimal solution to the mathematical programming problem (47) and x and y are components of the optimal solution,

$$max \sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^T (A^k + \grave{A}^k) y - \underline{a}^k x^T A^k y}{\left(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y\right)^2} + \lambda + \tau + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^T (B^l + \grave{B}^l) y - \underline{b}^l x^T B^l y\right)}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2}, + \acute{\lambda} + \acute{\tau}$$

subject to

$$\sum_{k=1}^{r} \vartheta_k \frac{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y) A^k y + (\overline{a}^k - x^T A^k y) \grave{A}^k y}{\left(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y\right)^2} + \lambda e^m + \tau K y \leq 0,$$

$$\sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right) x^T B^l + \left(\overline{b}^l - x^T B^l y\right) x^T \grave{B}^l}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda} e^n + \acute{\tau} x^T M \leq 0, \tag{47}$$

$$1 - x^T K y \leq 0,$$

$$1 - x^T M y \leq 0$$

$$e^{mT} x = 1,$$

$$e^{nT} y = 1,$$

$\acute{\tau}(1 - x^T K y) = 0,$
$\acute{t}(1 - x^T M y) = 0,$
$x \geq 0^m,$
$y \geq 0^n.$

Proof: The constraints of the optimization problem (47) are the constraints (37)-(46) of Kuhn-Tucker conditions. Assume that the feasible region of the above-mentioned problem is called S. In this region, according to (37) and (38), we have:

$$\sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^T(A^k + \grave{A}^k)y - \underline{a}^k x^T A^k y}{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y)^2} + \lambda + \tau + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^T(B^l + \grave{B}^l)y - \underline{b}^l x^T B^l y\right)}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda} + \acute{t}$$
$$= x^T \left\{ \sum_{k=1}^{r} \vartheta_k \frac{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y)A^k y + (\overline{a}^k - x^T A^k y)\grave{A}^k y}{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y)^2} + \lambda e^m + \tau K y \right\} \tag{48}$$
$$+ y\{\sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y)x^T B^l + (\overline{b}^l - x^T B^l y\right) x^T \grave{B}^l}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda} e^n + \acute{t} x^T M\} \leq 0$$

For

$$(x, y, \lambda, \acute{\lambda}, \tau, \acute{t}) \in S$$

Therefore,

$$max \sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^T(A^k + \grave{A}^k)y - \underline{a}^k x^T A^k y}{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y)^2} + \lambda + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^T(B^l + \grave{B}^l)y - \underline{b}^l x^T B^l y\right)}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda}, \leq 0 \tag{49}$$

Let $(x^*, y^*, \lambda^*, \acute{\lambda}^*, \tau^*, \acute{t}^*)$ satisfy Kuhn-Tucker conditions (35)-(46). Therefore, according to Eq. (35) and Eq. (36), we have:

$$\sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^{*T}(A^k + \grave{A}^k)y^* - \underline{a}^k x^{*T} A^k y^*}{(\overline{a}^k - \underline{a}^k + x^{*T} \grave{A}^k y^*)^2} + \lambda^* + \tau^* + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^{*T}(B^l + \grave{B}^l)y^* - \underline{b}^l x^{*T} B^l y^*\right)}{\left(\overline{b}^l - \underline{b}^l + x^{*T} \grave{B}^l y^*\right)^2} + \acute{\lambda}^* + \acute{t}^* = 0. \tag{50}$$

Based on Eq. (49), Eq. (50) and the fact $(x^*, y^*, \lambda^*, \acute{\lambda}^*, \tau^*, \acute{t}^*) \in S$,we have

$$\sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^{*T}(A^k + \grave{A}^k)y^* - \underline{a}^k x^{*T} A^k y^*}{(\overline{a}^k - \underline{a}^k + x^{*T} \grave{A}^k y^*)^2} + \lambda^* + \tau^* + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^{*T}(B^l + \grave{B}^l)y^* - \underline{b}^l x^{*T} B^l y^*\right)}{\left(\overline{b}^l - \underline{b}^l + x^{*T} \grave{B}^l y^*\right)^2} + \acute{\lambda}^* + \acute{t}^*$$
$$= \max_{x,y,\lambda,\acute{\lambda},\tau,\acute{t}} \sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^T(A^k + \grave{A}^k)y - \underline{a}^k x^T A^k y}{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y)^2} + \lambda + \tau + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^T(B^l + \grave{B}^l)y - \underline{b}^l x^T B^l y\right)}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda} + \acute{t} \tag{51}$$

Conversely, let $(x^*, y^*, \lambda^*, \acute{\lambda}^*, \tau^*, \acute{t}^*)$ *is* an optimal solution of the problem (47). Therefore, we have

$$\sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^{*T}(A^k + \grave{A}^k)y^* - \underline{a}^k x^{*T} A^k y^*}{(\overline{a}^k - \underline{a}^k + x^{*T} \grave{A}^k y^*)^2} + \lambda^* + \tau^* + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^{*T}(B^l + \grave{B}^l)y^* - \underline{b}^l x^{*T} B^l y^*\right)}{\left(\overline{b}^l - \underline{b}^l + x^{*T} \grave{B}^l y^*\right)^2} + \acute{\lambda}^* + \acute{t}^* \leq 0.$$

Considering the existence of the equilibrium solution and Kuhn-Tucker conditions (35)-(46), there exists at least one solution $(x, y, \lambda, \acute{\lambda}, \tau, \acute{t})$which satisfies

$$\sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^T(A^k + \grave{A}^k)y - \underline{a}^k x^T A^k y}{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y)^2} + \lambda + \tau + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^T(B^l + \grave{B}^l)y - \underline{b}^l x^T B^l y\right)}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda} + \acute{t} = 0$$

Therefore, if $(x^*, y^*, \lambda^*, \acute{\lambda}^*, \tau^*, \acute{t}^*)$ is the optimal solution of problem (47), there must be

$$\sum_{k=1}^{r} \vartheta_k \frac{\overline{a}^k x^{*T}(A^k + \grave{A}^k)y^* - \underline{a}^k x^{*T} A^k y^*}{(\overline{a}^k - \underline{a}^k + x^{*T} \grave{A}^k y^*)^2} + \lambda^* + \tau^* + \sum_{l=1}^{s} \omega_l \frac{\left(\overline{b}^l x^{*T}(B^l + \grave{B}^l)y^* - \underline{b}^l x^{*T} B^l y^*\right)}{\left(\overline{b}^l - \underline{b}^l + x^{*T} \grave{B}^l y^*\right)^2} + \acute{\lambda}^*$$
$$+ \acute{t}^* = 0.$$

According to the first and the second constraints of problem (47),

$$\sum_{k=1}^{r}\vartheta_k\frac{\overline{a}^k x^{*T}\left(A^k+\grave{A}^k\right)y^*-\underline{a}^k x^{*T}A^k y^*}{\left(\overline{a}^k-\underline{a}^k+x^{*T}\grave{A}^k y^*\right)^2}+\lambda^*+\tau^*=0,$$

$$\sum_{l=1}^{s}\omega_l\frac{\left(\overline{b}^l x^{*T}\left(B^l+\grave{B}^l\right)y^*-\underline{b}^l x^{*T}B^l y^*\right)}{\left(\overline{b}^l-\underline{b}^l+x^{*T}\grave{B}^l y^*\right)^2}+\acute{\lambda}^*+\acute{\tau}^*=0.$$

Hence $(x^*,y^*,\lambda^*,\acute{\lambda}^*,\tau^*,\acute{\tau}^*)$satisfies Kuhn-Tucker conditions.

Theorem 1. For a multiobjective bimatrix game$(\tilde{A},\tilde{B})$ with payoffs of Z numbers, let membership functions of the first part of payoffs and goals of Players I and II be linear. The necessary conditions for x and y to be an equilibrium solution with respect to the degree of attainment of the fuzzy goal aggregated by weighting coefficients is that x and y are components of an optimal solution to the mathematical programming problem (47).

Since the constraints of the problems (33) and (34) are linear and convex and the objective functions of them are concave, the necessary Kuhn-Tucker conditions (35)-(46) are also sufficient. Therefore, (x^*,y^*)obtained from the solution of problem (47) will be the optimal solution of problems (33) and (34) and consequently will be the equilibrium solution of bimatrix game with payoffs of z-numbers.

3.2. Aggregation by a minimum component

Consider an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component. This aggregation rule is often adopted in a multiple criteria decision making problem. Particularly in fuzzy decision making, this aggregation corresponds to the intersection of all of the fuzzy sets and a solution is determined by maximizing the degree of membership function of the intersection, and this decision rule is called Bellman and Zadeh's fuzzy decision rule.

Player I's fuzzy goals aggregated by a minimum component is represented as

$$Md_1(x,y)=\min_{k\in K}\{d_1^k(x,y)\} \tag{52}$$

The degree of attainment of the *l*th fuzzy goal for Player II can be defined in a similar way.

We now consider equilibrium solutions with respect to the degree of attainment of the aggregated fuzzy goal. A pair of strategies x^* and y^*,$(x^*,y^*)\in S$, is said to be an equilibrium solution if for any other mixed strategies *x* and *y*, $(x,y)\in S$,

$$\begin{aligned}&\min_{k\in K}\{d_1^k(x^*,y^*)\}\geq\min_{k\in K}\{d_1^k(x,y^*)\},\\&\min_{l\in L}\{d_2^l(x^*,y^*)\}\geq\min_{l\in L}\{d_2^l(x^*,y)\}.\end{aligned} \tag{53}$$

Employing aggregation by a minimum component, for a pair of x and y, Player I's degree of attainment of the aggregated fuzzy goal can be represented by

$$Md_1(x,y)=\min_{k\in K}\{d_1^k(x,y)\}=\min_{k\in K}\left\{\frac{x^T\left(A^k+\grave{A}^k\right)y-\underline{a}^k}{\overline{a}^k-\underline{a}^k+x^T\grave{A}^k y}\right\}=\min_{k\in K}\left\{\frac{\sum_{i=1}^m\sum_{j=1}^n(a_{ij}^k+\grave{a}_{ij}^k)x_i\,y_j-\underline{a}^k}{\overline{a}^k-\underline{a}^k+\sum_{i=1}^m\sum_{j=1}^n\grave{a}_{ij}^k\,x_i\,y_j}\right\}, \tag{54}$$

Player II's can be also represented similarly. A pair of optimal solutions $\boldsymbol{x}^*$and $\boldsymbol{y}^*$to the following two mathematical programming problems is an equilibrium solution.

$$\begin{aligned}&Md_1(x^*,y^*)=\max_{x,\sigma}\sigma\\&\text{subject to}\\&\frac{x^T\left(A^k+\grave{A}^k\right)y^*-\underline{a}^k}{\overline{a}^k-\underline{a}^k+x^T\grave{A}^k y^*}\geq\sigma,k=1,\dots,r,\end{aligned} \tag{55}$$

$$1 - x^T K y^* \leq 0,$$
$$e^{mT} x = 1,$$
$$x \geq 0^m,$$
$$Md_2(x^*, y^*) = \max_{y,\delta} \delta$$
subject to
$$\frac{x^{*T}(B^l + \grave{B}^l)y - \underline{b}^l}{\overline{b}^l - \underline{b}^l + x^{*T}\grave{B}^l y} \geq \delta, l = 1, \dots, s \tag{56}$$
$$1 - x^{*T} M y \leq 0,$$
$$e^{mT} y = 1,$$
$$y \geq 0^n.$$

Here σ and δ are auxiliary scalar variables. By applying the necessary conditions of Kuhn and Tucker to the problems (55) and (56), a necessary condition that a pair of x and y be an equilibrium solution with respect to the degree of attainment of the fuzzy goal becomes that there exist scalar values ψ *and* ξ ,an r-dimensional vector λ and an s-dimensional vector θ such that they satisfy:

$$\sum_{k=1}^{r} \theta_k \frac{\underline{a}^k\left(2x^T(A^k + \grave{A}^k)y - \overline{a}^k - \underline{a}^k\right) - x^T \grave{A}^k y x^T (A^k - \grave{A}^k) y}{\left(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y\right)^2} + \sigma + \lambda + \tau = 0, \tag{57}$$

$$\sum_{l=1}^{s} \acute{\theta}_l \frac{\underline{b}^l\left(2x^T(B^l + \grave{B}^l)y - \overline{b}^l - \underline{b}^l\right) - x^T \grave{B}^l y x^T (B^l - \grave{B}^l) y}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \delta + \acute{\lambda} + \acute{\tau} = 0, \tag{58}$$

$$\sum_{k=1}^{r} \theta_k \frac{(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y) A^k y + (\overline{a}^k - x^T A^k y) \grave{A}^k y}{\left(\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y\right)^2} + \lambda e^m + \tau K y \leq 0,$$

$$\sum_{l=1}^{s} \acute{\theta}_l \frac{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y) B^l y + (\overline{b}^l - x^T B^l y\right) \grave{B}^l y}{\left(\overline{b}^l - \underline{b}^l + x^T \grave{B}^l y\right)^2} + \acute{\lambda} e^n + \acute{\tau} x^T M \leq 0, \tag{59}$$

$$\frac{x^T(A^k + \grave{A}^k)y^* - \underline{a}^k}{\overline{a}^k - \underline{a}^k + x^T \grave{A}^k y^*} - \sigma \geq 0, k = 1, \dots, r \tag{60}$$

$$\frac{x^{*T}(B^l + \grave{B}^l)y - \underline{b}^l}{\overline{b}^l - \underline{b}^l + x^{*T}\grave{B}^l y} - \delta \geq 0, l = 1, \dots, s \tag{61}$$

$$1 - x^T K y \leq 0, \tag{62}$$
$$1 - x^T M y \leq 0 \tag{63}$$
$$e^{mT} x - 1 = 0, \tag{64}$$
$$e^{nT} y - 1 = 0, \tag{65}$$
$$\tau(1 - x^T K y) = 0, \tag{66}$$
$$\acute{\tau}(1 - x^T M y) = 0, \tag{67}$$
$$x \geq 0^m, \tag{68}$$
$$y \geq 0^n. \tag{69}$$

Here 0^r and 0^s are r-dimensional and s-dimensional vectors in which each of the entries is 0, respectively.
Theorem 2. For a multiobjective bimatrix game$(\tilde{A}, \tilde{B})$ with payoffs of Z numbers, let membership functions of the first part of payoffs and goals of Players I and II be linear. The necessary conditions for x and y to be an equilibrium solution with respect to the degree of attainment of the fuzzy goal aggregated

by a minimum component is that x and y are components of an optimal solution to the mathematical programming problem:

$$max\sum_{k=1}^{r}\theta_k\frac{\underline{a}^k\left(2x^T\left(A^k+\grave{A}^k\right)y-\overline{a}^k-\underline{a}^k\right)-x^T\grave{A}^kyx^T\left(A^k-\grave{A}^k\right)y}{\left(\overline{a}^k-\underline{a}^k+x^T\grave{A}^ky\right)^2}+\sigma+\lambda+\tau$$
$$+\sum_{l=1}^{s}\acute{\theta}_l\frac{\underline{b}^l\left(2x^T\left(B^l+\grave{B}^l\right)y-\overline{b}^l-\underline{b}^l\right)-x^T\grave{B}^lyx^T\left(B^l-\grave{B}^l\right)y}{\left(\overline{b}^l-\underline{b}^l+x^T\grave{B}^ly\right)^2}+\delta+\acute{\lambda}+\acute{\tau},$$

$$\sum_{k=1}^{r}\theta_k\frac{\left(\overline{a}^k-\underline{a}^k+x^T\grave{A}^ky\right)A^ky+\left(\overline{a}^k-x^TA^ky\right)\grave{A}^ky}{\left(\overline{a}^k-\underline{a}^k+x^T\grave{A}^ky\right)^2}+\lambda e^m+\tau Ky\leq 0,$$

$$\sum_{l=1}^{s}\acute{\theta}_l\frac{\left(\overline{b}^l-\underline{b}^l+x^T\grave{B}^ly\right)B^ly+\left(\overline{b}^l-x^TB^ly\right)\grave{B}^ly}{\left(\overline{b}^l-\underline{b}^l+x^T\grave{B}^ly\right)^2}+\acute{\lambda}e^n+\acute{\tau}x^TM\leq 0,$$

$$\frac{x^T\left(A^k+\grave{A}^k\right)y^*-\underline{a}^k}{\overline{a}^k-\underline{a}^k+x^T\grave{A}^ky^*}-\sigma\geq 0, k=1,\ldots,r$$

$$\frac{x^{*T}\left(B^l+\grave{B}^l\right)y-\underline{b}^l}{\overline{b}^l-\underline{b}^l+x^{*T}\grave{B}^ly}-\delta\geq 0, l=1,\ldots,s \qquad (70)$$

$$1-x^TKy\leq 0,$$

$$1-x^TMy\leq 0$$

$$e^{mT}x-1=0,$$

$$e^{nT}y-1=0,$$

$$\tau(1-x^TKy)=0,$$

$$\acute{\tau}(1-x^TMy)=0,$$

$$x\geq 0^m,$$

$$y\geq 0^n.$$

Theorem 2 can be proven by using a lemma similar to Lemma 1. If $d_1(x,y)$ and $d_2(x^*,y^*)$ are concave with respect to x and y, respectively, it is easily verified that $Md_1(x,y)$and $Md_2(x^*,y^*)$are concave with respect to x and y, respectively. Then, Theorem 2 gives the necessary and sufficient conditions.

Numerical Example –Consider the multiobjective bimatrix game with payoffs of z-numbers:

$$A^1=\begin{bmatrix}((100,10,15),(0.7,0.05,0.05)) & ((85,5,10),(0.9,0.04,0.05))\\ ((80,5,5),(0.8,0.03,0.02)) & ((185,15,20),(0.9,0.1,0.08))\end{bmatrix},$$
$$A^2=\begin{bmatrix}((30,2,5),(0.8,0.05,0.05)) & ((25,5,3),(0.95,0.05,0.1))\\ ((40,5,5),(0.9,0.04,0.02)) & ((35,5,4),(0.85,0.1,0.08))\end{bmatrix},$$
$$B^1=\begin{bmatrix}((140,5,15),(0.75,0.05,0.1)) & ((100,10,10),(0.9,0.1,0.05))\\ ((120,10,5),(0.65,0.03,0.05)) & ((135,5,10),(0.85,0.05,0.05))\end{bmatrix},$$
$$B^2=\begin{bmatrix}((30,5,5),(0.8,0.05,0.05)) & ((35,5,10),(0.9,0.1,0.05))\\ ((40,10,5),(0.85,0.05,0.05)) & ((25,5,10),(0.9,0.1,0.05))\end{bmatrix},$$

The level of expected confidence for the first and the second players is considered about 80%, $\tilde{R}_1=$ (0.8,0.05,0.05) and 75%, $\tilde{R}_2=$ (0.75,0.05,0.05), respectively.
Goal parameters including $\underline{a}$ and $\overline{a}$ are calculated based on Eq. (22) and Eq. (23) as follows,

$$\underline{a}^1=75, \underline{a}^2=22.5, \bar{a}^1=205, \bar{a}^2=45 \qquad \underline{b}^1=90, \underline{b}^2=20, \bar{b}^1=57, \bar{b}^2=155.$$

We employ two basic methods, one by weighting coefficients and the other by a minimum component to solving problem. Let player I's and II's weighting coefficients for fuzzy goals in first method be: $\vartheta = (0.7, 0.3)$, and $\omega = (0.4, 0.6)$, respectively. The equilibrium solution of the above-mentioned problem is obtained through solving the mathematical programming problems (47) and (70) and the results are given in Table 1 as follows,

Table 1
The equilibrium solution of the proposed study

Method	Weighting coefficients	Minimum component
x	$(0, 1)$	$(0.467, 0.533)$
y	$(0.517, 0.483)$	$(0.271, 0.729)$

In addition, the fuzzy values of the game for the two players are summarized in Table 2 as follows,

Table 2
The results of fuzzy values of the game for the two players

Method	Weighting coefficients	Minimum component
$\tilde{E}_1^1(x, y)$	$(130.72, 9.83, 12.245)$	$(125.03, 9.518, 13.796)$
$\tilde{E}_1^2(x, y)$	$(32.755, 2.483, 1.449)$	$(31.204, 1.921, 2.479)$
$\tilde{E}_2^1(x, y)$	$(127.25, 7.585, 7.415)$	$(121.55, 7.424, 9.911)$
$\tilde{E}_2^2(x, y)$	$(47.755, 2.932, 3.449)$	$(37.933, 3.109, 3.633)$

In addition, the levels of achievement to the goals for the first and the second players are summarized in Table 3 as follows,

Table 3
The levels of achievement to the goals for the first and the second players

Method	Weighting coefficients	Minimum component
$\tilde{d}_1^1(x, y)$	0.4778	0.4439
$\tilde{d}_1^2(x, y)$	0.4992	0.4585
$\tilde{d}_2^1(x, y)$	0.6167	0.5535
$\tilde{d}_2^2(x, y)$	0.7714	0.5308

In addition, the level of confidence of them to the solution are given in Table 4.

Table 4
The level of confidence

Method	Weighting coefficients	Minimum component
$\widetilde{Cr}_1$	$(0.8242, 0.0638, 0.049)$	$(0.841, 0.063, 0.057)$
$\widetilde{Cr}_2$	$(0.7466, 0.0397, 0.05)$	$(0.825, 0.064, 0.056)$

4. Conclusions

The present study investigated the equilibrium solution for non-cooperative multiobjective bimatrix game with payoffs and goals of z-numbers. The equilibrium solution was defined based on the level of achievement to the aggregated goal of z-number. Then two nonlinear programming problems were developed using the objective function of the level of achievement to aggregated goal and the constraint of the confidence level for each player. We have shown the sufficient and necessary conditions that pair of strategies be the equilibrium solution and an optimization model was developed to find them. Finally, an illustrative example is given in order to show the detailed calculation process.

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