An inventory model for perishable items with quadratic trapezoidal type demand under partial backlogging

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CHRONICLE	ABSTRACT
	In this paper, we consider the inventory model for perishable items with quadratic trapezoidal type demand rate, that is, the demand rate is a piecewise quadratic function under constant deterioration rate. The model consider allows for shortages and the demand is partially backlogged. The model is solved analytically by minimizing the total inventory cost. The result is illustrated with numerical example. Finally, we discuss sensitivity analysis for the model.

1. Introduction

Deteriorating items are very common issue in our daily life circumstances. In recent years, many researchers have studied inventory models for deteriorating items, however, academia has not reached a consensus on the definition of the deteriorating items. According to Wee (1993), deteriorating items refers to the items that become decayed, damaged, evaporative, expired, invalid, devaluation and so on through time. According to the definition, deteriorating items can be classified into two categories. The first category refers to the items that become decayed, damaged, evaporative, or expired through time, like meat, vegetables, fruit, medicine, flowers and so on; the other category refers to the items that lose part or total value through time because of new technology or the introduction of alternatives, like computer chips, mobile phones, fashion and seasonal goods and so on. The inventory problem of deteriorating items was first studied by Whitin (1957), he studied fashion items deteriorating at the end of the storage period. Then, Ghare and Schrader (1963) concluded in their study that the consumption of the deteriorating items was closely relative to a negative exponential function of time. Various authors

such as Deng et al. (2007), Cheng and Wang (2009), Cheng et al. (2011) and Hung (2011) studied inventory models for deteriorating items in various aspects.

In world business market, demand has been always one of the most key factors in the decisions relating to the inventory and production activities. There are mainly two categories demands in the present studies, one is deterministic demand and the other is stochastic demand. Various formations of consumption tendency have been studied, such as constant demand (Padmanabhan & Vrat, 1990; Chung & Lin, 2001; Benkherouf et al., 2003; Chu et al., 2004), level-dependent demand (Giri & Choudhuri, 1998; Chung et al., 2000; Bhattacharya, 2005; Wu et al., 2006), price dependent demand (Wee & Law, 1999; Abad, 1996, 2001), time dependent demand (Resh et al., 1976; Henery, 1979; Sachan, 1984; Dave, 1989; Teng, 1996; Teng et al., 2002; Skouri & Papachristos, 2002; Panda et al., 2012; Sett et al., 2013; Mishra et al., 2013) and time and price dependent demand (Wee, 1995). Among them, ramp type demand is a special type of time dependent demand. Hill (1995), one of the pioneers, developed an inventory model with ramp type demand that begins with a linear increasing demand until to the turning point, denoted as μ, proposed by previous researchers, then it becomes a constant demand. There has been a movement towards developing this type of inventory system for minimum cost and maximum profit problems. Several authors: Mandal and Pal (1998) focused on deteriorating items. Wu et al. (1999) were concerned with backlog rates relative to the waiting time. Wu and Ouyang (2000) tried to build an inventory system under two replenishment policies: starting with shortage or without shortage. Wu (2001) considered the deteriorated items satisfying Weibull distribution. Giri et al (2003) dealt with more generalized three parameter Weibull deterioration distribution. Deng (2005) extended the inventory model of Wu et al. (1999) for the situation where the in-stock period is shorter than μ. Manna and Chaudhuri (2006) set up a model where the deterioration is dependent on time. Panda et al. (2007) constructed an inventory model with a comprehensive ramp type demand. Deng et al. (2007) contributed to the revision of Mandal and Pal (1998), and Wu and Ouyang (2000). Panda et al. (2008) examined the cyclic deterioration items. Wu et al. (2008) studied the maximum profit problem with the stock-dependent selling rate. They developed two inventory models all related to the conversion of the ramp type demand, and then examined the optimal solution for each case. However, in a realistic product life cycle, demand is increasing with time during the growth phase. Then, after reaching its peak, the demand becomes stable for a finite time period called the maturity phase. Thereafter, the demand starts decreasing with time and eventually reaching zero or constant.

In this work, we extend Hill's ramp type demand rate to quadratic trapezoidal type demand rate. Such type of demand pattern is generally seen in the case of any fad or seasonal goods coming to market. The demand rate for such items increases quadratic-ally with the time up to certain time and then ultimately stabilizes and becomes constant, and finally the demand rate approximately decreases to a constant, and then begins the next replenishment cycle. We think that such type of demand rate is quite natural and useful in real world market situation. One can think that our work may provide a solid foundation for the future study of this kind of important inventory models with quadratic trapezoidal type demand rate.

2. Assumption and notations

The fundamental assumption and notations used in this paper are given as follows:

(1) The demand rate, $R(t)$, which is positive and consecutive, is assumed to be a quadratic trapezoidal type function of time, that is

$$R(t) = \begin{cases} a_1 + b_1 t + c_1 t^2, & t \leq \mu_1, \\ R_0, & \mu_1 \leq t \leq \mu_2, \\ a_2 + b_2 t - c_2 t^2, & \mu_2 \leq t \leq T \end{cases} \tag{1}$$

Chose a_1, b_1, c_1, a_2, b_2 and c_2 such a way that $a_2 + b_2 t - c_2 t^2$ should not be negative for $\mu_2 \leq t \leq T$. where μ_1 is the time point changing from the increasing quadratic demand to constant demand, and μ_2 is the time point changing from the constant demand to the decreasing demand.

(2)   Replenishment rate is infinite, thus replenishment is instantaneous.
(3) $I(t)$ is the inventory level at any time t, $0 \leq t \leq T$.
(4) T is the fixed length of each ordering cycle.
(5) θ is the constant rate of deterioration, $0 < \theta < 1$.
(6) t_1 is the time when the inventory level reaches zero.
(7) $t_1{}^*$ is an optimal point.
(8) k_0 is the fixed ordering cost per order.
(9) k_1 is the cost of each deteriorated item.
(10) k_2 is the inventory holding cost per unit per unit of time.
(11) k_3 is the shortage cost per unit per unit of time.
(12) S is the maximum inventory level for the ordering cycle, such that $S=I(0)$.
(13) Q is the ordering quantity per cycle.
(14) $A_1(t_1)$ is the average total cost per unit time under the condition $t_1 \leq \mu_1$.
(15) $A_2(t_1)$ is the average total cost per unit time, for $\mu_1 \leq t_1 \leq \mu_2$.
(16) $A_3(t_1)$ is the average total cost per unit time, for $\mu_2 \leq t_1 \leq T$.

3. Mathematical and theoretical results

Here, we consider the deteriorating inventory model where demand rate is trapezoidal type quadratic function. Replenishment occurs at time $t=0$ when the inventory level attains its maximum. For $t \in [0, t_1]$, the inventory level reduces due to both demand and deterioration. At time t_1, the inventory level reaches zero, then shortage is allowed to occur during the interval (t_1, T), and all of the demand during the shortage period (t_1, T) is completely backlogged. The total amount of backlogged items is replaced by the next replenishment. The rate of change of the inventory during the stock period $[0, t_1]$ and shortage period (t_1, T) is governed by the following differential equations:

$$\frac{dI(t)}{dt} + \theta I(t) + R(t) = 0 , \ 0 < t < t_1, \tag{2}$$

$$\frac{dI(t)}{dt} + R(t) = 0 , \ t_1 < t < T , \tag{3}$$

with boundary condition $I(0)=S$ and $I(t_1)=0$. One can think about t_1, t_1 may occur within $[0, \mu_1]$ or $[\mu_1, \mu_2]$ or $[\mu_2, T]$. Hence in this paper we are going to discuss all three possible cases.

Case 1: $0 < t_1 \leq \mu_1$

The quadratic trapezoidal type market demand and constant rate of deterioration, the inventory level gradually diminishes during the period $[0, t_1]$ and ultimately reaches to zero at time $t=t_1$. Then, from Eq. (2) and Eq. (3), we have

$$\frac{dI(t)}{dt} + \theta I(t) + a_1 + b_1 t + c_1 t^2 = 0 , \ 0 < t < t_1 \tag{4}$$

$$\frac{dI(t)}{dt} + a_1 + b_1 t + c_1 t^2 = 0 , \ t_1 < t < \mu_1 \tag{5}$$

$$\frac{dI(t)}{dt} + R_0 = 0, \quad \mu_1 < t < \mu_2 \tag{6}$$

$$\frac{dI(t)}{dt} + a_2 + b_2 t - c_2 t^2 = 0, \quad \mu_2 < t < T \tag{7}$$

Now solving the differential Eqs. (4-7) with the condition $I(t_1)=0$ and continuous property of $I(t)$, we get

$$I(t) = \left(\frac{a_1 + b_1 t_1 + c_1 t_1^2}{\theta} - \frac{b_1 + 2c_1 t_1}{\theta^2} + \frac{2c_1}{\theta^3} \right) e^{\theta(t_1 - t)} - \frac{a_1 + b_1 t + c_1 t^2}{\theta} + \frac{b_1 + 2c_1 t}{\theta^2} - \frac{2c_1}{\theta^3}, \quad 0 \le t \le t_1 \tag{8}$$

$$I(t) = (t_1 - t)a_1 + (t_1^2 - t^2)\frac{b_1}{2} + (t_1^3 - t^3)\frac{c_1}{3}, \quad t_1 \le t \le \mu_1 \tag{9}$$

$$I(t) = -R_0 t + a_1 t_1 + (t_1^2 + \mu_1^2)\frac{b_1}{2} + (t_1^3 + 2\mu_1^3)\frac{c_1}{3}, \quad \mu_1 \le t \le \mu_2 \tag{10}$$

$$I(t) = a_1 t_1 - a_2 t + (t_1^2 + \mu_1^2)\frac{b_1}{2} + (t_1^3 + 2\mu_1^3)\frac{c_1}{3} - (t^2 + \mu_2^2)\frac{b_2}{2} + (t^3 + 2\mu_2^3)\frac{c_2}{3}, \quad \mu_2 \le t \le T \tag{11}$$

The beginning inventory level can be computed as

$$S = I(0) = \left(\frac{a_1}{\theta} - \frac{b_1}{\theta^2} + \frac{2c_1}{\theta^3} \right)(e^{\theta t_1} - 1) + \left(\frac{b_1 t_1 + c_1 t_1^2}{\theta} - \frac{2c_1 t_1}{\theta^2} \right)e^{\theta t_1} \tag{12}$$

The total number of items which is perish in the interval $[0, t_1]$, say D_T, is

$$D_T = S - \int_0^{t_1} R(t)dt = S - \int_0^{t_1} (a_1 + b_1 t + c_1 t^2)dt$$

$$= \left(\frac{a_1}{\theta} - \frac{b_1}{\theta^2} + \frac{2c_1}{\theta^3} \right)(e^{\theta t_1} - 1) + \left(\frac{b_1 t_1 + c_1 t_1^2}{\theta} - \frac{2c_1 t_1}{\theta^2} \right)e^{\theta t_1} - a_1 t_1 - \frac{b_1 t_1^2}{2} - \frac{c_1 t_1^3}{3} \tag{13}$$

The total amounts of inventory carried during the interval $[0, t_1]$, say C_T, is

$$C_T = \int_0^{t_1} I(t)dt = \int_0^{t_1} \left[\begin{array}{l} \left(\dfrac{a_1 + b_1 t_1 + c_1 t_1^2}{\theta} - \dfrac{b_1 + 2c_1 t_1}{\theta^2} + \dfrac{2c_1}{\theta^3} \right)e^{\theta(t_1 - t)} \\ -\dfrac{a_1 + b_1 t + c_1 t^2}{\theta} + \dfrac{b_1 + 2c_1 t}{\theta^2} - \dfrac{2c_1}{\theta^3} \end{array} \right] dt$$

$$= \left(\frac{a_1 + b_1 t_1 + c_1 t_1^2}{\theta^2} - \frac{b_1 + 2c_1 t_1}{\theta^3} + \frac{2c_1}{\theta^4} \right)(e^{\theta t_1} - 1) + \left(\frac{b_1}{\theta^2} - \frac{a_1}{\theta} - \frac{2c_1}{\theta^3} \right)t_1 + \left(\frac{c_1}{\theta^2} - \frac{b_1}{2\theta} \right)t_1^2 - \frac{c_1}{3\theta}t_1^3 \tag{14}$$

The total shortage quantity during the interval $[t_1, T]$, say B_T, is

$$B_T = -\int_{t_1}^{T} I(t)dt = -\int_{t_1}^{\mu_1} I(t)dt - \int_{\mu_1}^{\mu_2} I(t)dt - \int_{\mu_2}^{T} I(t)dt = -\int_{t_1}^{\mu_1} \left[(t_1 - t)a_1 + (t_1^2 - t^2)\frac{b_1}{2} + (t_1^3 - t^3)\frac{c_1}{3} \right]dt$$

$$-\int_{\mu_1}^{\mu_2} \left[-R_0 t + a_1 t_1 + (t_1^2 + \mu_1^2)\frac{b_1}{2} + (t_1^3 + 2\mu_1^3)\frac{c_1}{3} \right]dt$$

$$-\int_{\mu_2}^{T}\left[a_1t_1 - a_2t + (t_1^2 + \mu_1^2)\frac{b_1}{2} + (t_1^3 + 2\mu_1^3)\frac{c_1}{3} - (t^2 + \mu_2^2)\frac{b_2}{2} + (t^3 + 2\mu_2^3)\frac{c_2}{3}\right]dt \tag{15}$$

$$
\begin{aligned}
= &-(t_1\mu_1 - t_1^2)a_1 + \frac{a_1}{2}(\mu_1^2 - t_1^2) - \frac{b_1}{2}t_1^2(\mu_1 - t_1) + \frac{b_1}{6}(\mu_1^3 - t_1^3) - \frac{c_1}{3}t_1^3(\mu_1 - t_1) + \frac{c_1}{12}(\mu_1^4 - t_1^4) \\
&+ \frac{R_0}{2}(\mu_1^2 - \mu_2^2) - a_1t_1(\mu_2 - \mu_1) - \frac{b_1}{2}t_1^2(\mu_2 - \mu_1) - \frac{b_1}{2}\mu_1^2(\mu_2 - \mu_1) - \frac{c_1}{3}t_1^3(\mu_2 - \mu_1) \\
&- \frac{2c_1}{3}\mu_1^3(\mu_2 - \mu_1) - a_1t_1(T - \mu_2) + \frac{a_2}{2}(T^2 - \mu_2^2) - \frac{b_1}{2}t_1^2(T - \mu_2) - \frac{b_1}{2}\mu_1^2(T - \mu_2) \\
&- \frac{c_1}{3}(t_1^3 + 2\mu_1^3)(T - \mu_2) + \frac{b_2}{6}(T^3 - \mu_2^3) + \frac{b_2}{2}\mu_2^2(T - \mu_2) - \frac{c_2}{12}(T^4 - \mu_2^4) - \frac{2c_2}{3}\mu_2^3(T - \mu_2)
\end{aligned}
$$

The average total cost per unit time for $0 < t_1 \le \mu_1$ is given by

$$A_1(t_1) = \frac{1}{T}[k_0 + k_1 D_T + k_2 C_T + k_3 B_T] \tag{16}$$

The first order derivative of $A_1(t_1)$ with respect to t_1 is as follows:

$$\frac{dA_1(t_1)}{dt_1} = \frac{1}{T}\left[\left(k_1 + \frac{k_2}{\theta}\right)(e^{\theta t_1} - 1) + k_3(t_1 - T)\right](a_1 + b_1t_1 + c_1t_1^2) \tag{17}$$

The necessary condition for $A_1(t_1)$ to be minimized, is $\dfrac{dA_1(t_1)}{dt_1} = 0$, that is

$$\frac{1}{T}\left[\left(k_1 + \frac{k_2}{\theta}\right)(e^{\theta t_1} - 1) + k_3(t_1 - T)\right](a_1 + b_1t_1 + c_1t_1^2) = 0 \tag{18}$$

This implies that

$$\left[\left(k_1 + \frac{k_2}{\theta}\right)(e^{\theta t_1} - 1) + k_3(t_1 - T)\right] = 0 \tag{19}$$

Let $p(t_1) = \left(k_1 + \dfrac{k_2}{\theta}\right)(e^{\theta t_1} - 1) + k_3(t_1 - T)$, \hfill (20)

Since $p(0) = -k_3T < 0$, $p(T) = \left(k_1 + \dfrac{k_2}{\theta}\right)(e^{\theta T} - 1) > 0$ and $p'(t_1) = \left(k_1 + \dfrac{k_1}{\theta}\right)e^{\theta t_1}\theta + k_3 > 0$, it implies that

$p(t_1)$ is a strictly monotonically increasing function and Eq. (19) has a unique solution at t_1^*, for $t_1^* \in (0, T)$. Therefore, we have

Property-1

The constant deteriorating rate of an inventory model with quadratic trapezoidal type demand rate under the time interval $0 < t_1 \le \mu_1$, $A_1(t_1)$ attains its minimum at $t_1 = t_1^*$, where $p(t_1^*) = 0$ if $t_1^* < \mu_1$. On the other hand, $A_1(t_1)$ attains its minimum at $t_1^* = \mu_1$ if $t_1^* \ge \mu_1$.

The total back order amount at the end of the cycle is follows,

$$\Delta_1 = -a_1 t_1^* + a_2 T - \frac{b_1}{2}(t_1^{*2} + \mu_1^2) - \frac{c_1}{3}(t_1^{*3} + 2\mu_1^3) + \frac{b_2}{2}(T^2 + \mu_2^2) - \frac{c_2}{3}(T^3 + 2\mu_2^3)$$ (21)

Therefore, the optimal order quantity, denoted by Q^*, is $Q^* = S^* + \Delta_1$, where S^* denote the optimal value of S.

Case-II, $\mu_1 \le t_1 \le \mu_2$

For the time period $t_1 \in [\mu_1, \mu_2]$, then, the differential equations governing the inventory model can be expressed as follows:

$$\frac{dI(t)}{dt} + \theta I(t) + a_1 + b_1 t + c_1 t^2 = 0 , \ 0 < t < \mu_1$$ (22)

$$\frac{dI(t)}{dt} + \theta I(t) + R_0 = 0 , \ \mu_1 < t < t_1$$ (23)

$$\frac{dI(t)}{dt} + R_0 = 0 , \quad t_1 < t < \mu_2$$ (24)

$$\frac{dI(t)}{dt} + a_2 + b_2 t - c_2 t^2 = 0 , \ \mu_2 < t < T$$ (25)

Solving differential Eq. (22-25), using $I(t_1)=0$, we get

$$I(t) = \left(\frac{R_0}{\theta} e^{\theta t_1} - \frac{b_1}{\theta^2} e^{\theta \mu_1} \right) e^{-\theta t} - \frac{a_1 + b_1 t + c_1 t^2}{\theta} + \frac{b_1 + 2 c_1 t}{\theta^2} - \frac{2 c_1}{\theta^3} - \frac{2 c_1 \mu_1}{\theta^2} e^{\theta(\mu_1 - t)} + \frac{2 c_1}{\theta^3} e^{\theta(\mu_1 - t)}, \ 0 \le t \le \mu_1$$ (26)

$$I(t) = \frac{R_0}{\theta}(e^{\theta(t_1 - t)} - 1) , \ \mu_1 \le t \le t_1$$ (27)

$$I(t) = R_0(t_1 - t), \ t_1 \le t \le \mu_2$$ (28)

$$I(t) = R_0 t_1 - a_2 t - \frac{b_2}{2}(t^2 + \mu_2^2) + \frac{c_2}{3}(t^3 + 2\mu_2^3), \ \mu_2 \le t \le T$$ (29)

The beginning inventory can be computed as

$$S = I(0) = \frac{R_0}{\theta} e^{\theta t_1} - \frac{b_1}{\theta^2} e^{\theta \mu_1} - \frac{a_1}{\theta} + \frac{b_1}{\theta^2} - \frac{2 c_1}{\theta^3} - \frac{2 c_1 \mu_1}{\theta^2} e^{\theta \mu_1} + \frac{2 c_1}{\theta^3} e^{\theta \mu_1}$$ (30)

The total amount of items which is perish within the time interval $[0, t_1]$ is

$$D_T = S - \int_0^{t_1} R(t) dt = S - \int_0^{\mu_1} (a_1 + b_1 t + c_1 t^2) dt - \int_{\mu_1}^{t_1} R_0 dt = \frac{R_0}{\theta} e^{\theta t_1} - \left(\frac{b_1}{\theta^2} + \frac{2 c_1 \mu_1}{\theta^2} - \frac{2 c_1}{\theta^3} \right) e^{\theta \mu_1}$$

$$-R_0(t_1 - \mu_1) - a_1(\mu_1 + \frac{1}{\theta}) + b_1 \left(\frac{1}{\theta^2} - \frac{\mu_1^2}{2} \right) - c_1 \left(\frac{2}{\theta^3} + \frac{\mu_1^3}{3} \right)$$ (31)

The total amount of inventory carried during the time interval $[0, t_1]$ is

$$C_T = \int_0^{t_1} I(t)dt = \int_0^{\mu_1} I(t)dt + \int_{\mu_1}^{t_1} I(t)dt \tag{32}$$

$$= \int_0^{\mu_1} \left[\left(\frac{R_0}{\theta} e^{\theta t_1} - \frac{b_1}{\theta^2} e^{\theta \mu_1} \right) e^{-\theta t} - \frac{a_1 + b_1 t + c_1 t^2}{\theta} + \frac{b_1 + 2c_1 t}{\theta^2} - \frac{2c_1}{\theta^3} \right] dt + \int_{\mu_1}^{t_1} \left[\frac{R_0}{\theta} (e^{\theta(t_1 - t)} - 1) \right] dt$$

$$- \frac{2c_1 \mu_1}{\theta^2} e^{\theta(\mu_1 - t)} + \frac{2c_1}{\theta^3} e^{\theta(\mu_1 - t)}$$

$$= \frac{b_1}{\theta^3} + \frac{R_0}{\theta^2} e^{\theta t_1} - \frac{b_1}{\theta^3} e^{\theta \mu_1} - \frac{1}{\theta} \left(\theta \mu_1 + \frac{b_1 \mu_1^2}{2} + \frac{c_1 \mu_1^3}{3} \right) + \frac{b_1 \mu_1}{\theta^2} - \frac{2c_1 \mu_1}{\theta^3} e^{\theta \mu_1}$$

$$- \frac{2c_1}{\theta^4} + \frac{2c_1}{\theta^4} e^{\theta \mu_1} - \frac{R_0}{\theta^2} - \frac{R_0}{\theta} (t_1 - \mu_1)$$

The total amount of shortage during the interval $[t_1, T]$

$$B_T = -\int_{t_1}^{T} I(t)dt = -\int_{t_1}^{\mu_2} I(t)dt - \int_{\mu_2}^{T} I(t)dt$$

$$= -\int_{t_1}^{\mu_2} R_0(t_1 - t)dt - \int_{\mu_2}^{T} \left[R_0 t_1 - a_2 t - \frac{b_2}{2}(t^2 + \mu_2^2) + \frac{c_2}{3}(t^3 + 2\mu_2^3) \right] dt \tag{33}$$

$$= -R_0 t_1 (\mu_2 - t_1) + \frac{R_0}{2}(\mu_2^2 - t_1^2) - R_0 t_1 (T - \mu_2) + \frac{a_2}{2}(T^2 - \mu_2^2) + \frac{b_2}{6}(T^3 - \mu_2^3)$$

$$+ \frac{b_2}{2}\mu_2^2(T - \mu_2) - \frac{c_2}{12}(T^4 - \mu_2^4) - \frac{2c_2}{3}\mu_2^3(T - \mu_2)$$

Now, the average total cost per unit time under the condition $\mu_1 \le t_1 \le \mu_2$, can be obtained as

$$A_2(t_1) = \frac{1}{T}[k_0 + k_1 D_T + k_2 C_T + k_3 B_T] \tag{34}$$

The first order derivative of $A_2(t_1)$ with respect to t_1 is given by

$$\frac{dA_2(t_1)}{dt_1} = \frac{R_0}{T} \left[\left(k_1 + \frac{k_2}{\theta} \right)(e^{\theta t_1} - 1) + k_3(t_1 - T) \right] \tag{35}$$

The required necessary condition for $A_2(t_1)$ to be minimized is $\dfrac{dA_2(t_1)}{dt_1} = 0$, that is

$$\left[\left(k_1 + \frac{k_2}{\theta} \right)(e^{\theta t_1} - 1) + k_3(t_1 - T) \right] = 0 \tag{36}$$

Let $p(t_1) = \left(k_1 + \dfrac{k_2}{\theta} \right)(e^{\theta t_1} - 1) + k_3(t_1 - T)$, \tag{37}

since $p'(t_1) = \left(k_1 + \dfrac{k_1}{\theta} \right) e^{\theta t_1} \theta + k_3 > 0$, which implies that $p(t_1)$ is strictly monotonically increasing function during the interval $\mu_1 \le t_1 \le \mu_2$.

Property-2

The constant deteriorating rate of an inventory model with quadratic trapezoidal type demand function during the time interval $\mu_1 \le t_1 \le \mu_2$, $A_2(t_1)$ attains its minimum at $t_1^* = \mu_1$ if $t_1^* < \mu_1$ and $A_2(t_1)$ attains its minimum at $t_1^* = \mu_2$ if $\mu_2 < t_1^*$.

Now, we can calculate the total amount of back-order quantity at the end of the cycle is

$$\Delta_2 = -R_0 t_1^* + a_2 T + \frac{b_2}{2}(T^2 + \mu_2^2) - \frac{c_2}{3}(T^3 + 2\mu_2^3) \tag{38}$$

Therefore, the optimal order quantity denoted by Q^* is $Q^* = S^* + \Delta_2$, where S^* denotes the optimal vale of S.

Case-III $\mu_2 \le t_1 < T$

For the time interval $t_1 \in [\mu_2, T)$, then, the differential equations governing the inventory model can be expressed as follows:

$$\frac{dI(t)}{dt} + \theta I(t) + a_1 + b_1 t + c_1 t^2 = 0, \ 0 < t < \mu_1 \tag{39}$$

$$\frac{dI(t)}{dt} + \theta I(t) + R_0 = 0, \ \mu_1 < t < \mu_2 \tag{40}$$

$$\frac{dI(t)}{dt} + \theta I(t) + a_2 + b_2 t - c_2 t^2 = 0, \ \mu_2 < t < t_1 \tag{41}$$

$$\frac{dI(t)}{dt} + a_2 + b_2 t - c_2 t^2 = 0, \ t_1 < t < T \tag{42}$$

Solving the differential Eqs. (39-42) with $I(t_1)=0$, we can get

$$I(t) = \frac{b_1 + 2c_1 t}{\theta^2} - \frac{2c_1}{\theta^3} - \frac{a_1 + b_1 t + c_1 t^2}{\theta} + \left(\frac{a_2 + b_2 t_1 - c_2 t_1^2}{\theta} - \frac{b_2}{\theta^2} + \frac{2c_2 t_1}{\theta^2} - \frac{2c_2}{\theta^3}\right)e^{\theta(t_1 - t)} \tag{43}$$

$$+ \left(\frac{b_2 - 2c_2\mu_2}{\theta^2} + \frac{2c_2}{\theta^3}\right)e^{\theta(\mu_2 - t)} + \left(\frac{2c_1}{\theta^3} - \frac{b_1 + 2c_1\mu_1}{\theta^2}\right)e^{\theta(\mu_1 - t)}, \ 0 \le t \le \mu_1$$

$$I(t) = -\frac{R_0}{\theta} + \left(\frac{a_2 + b_2 t_1 - c_2 t_1^2}{\theta} - \frac{b_2}{\theta^2} + \frac{2c_2 t_1}{\theta^2} - \frac{2c_2}{\theta^3}\right)e^{\theta(t_1 - t)} + \left(\frac{b_2 - 2c_2\mu_2}{\theta^2} + \frac{2c_2}{\theta^3}\right)e^{\theta(\mu_2 - t)} \tag{44}$$

$$\mu_1 \le t \le \mu_2$$

$$I(t) = \left(\frac{a_2 + b_2 t_1 - c_2 t_1^2}{\theta} - \frac{b_2}{\theta^2} + \frac{2c_2 t_1}{\theta^2} - \frac{2c_2}{\theta^3}\right)e^{\theta(t_1 - t)} + \frac{b_2 - 2c_2 t}{\theta^2} + \frac{2c_2}{\theta^3} - \frac{a_2 + b_2 t - c_2 t^2}{\theta}, \tag{45}$$

$$\mu_2 \le t \le t_1$$

$$I(t) = a_2(t_1 - t) + \frac{b_2}{2}(t_1^2 - t^2) + \frac{c_2}{3}(t^3 - t_1^3), \ t_1 \le t \le T \tag{46}$$

The total amount of inventory level at the beginning can be computed as

$$S = I(0) = \frac{b_1}{\theta^2} - \frac{2c_1}{\theta^3} - \frac{a_1}{\theta} + \left(\frac{a_2 + b_2 t_1 - c_2 t_1^2}{\theta} - \frac{b_2}{\theta^2} + \frac{2c_2 t_1}{\theta^2} - \frac{2c_2}{\theta^3}\right)e^{\theta t_1}$$

$$+ \left(\frac{b_2 - 2c_2\mu_2}{\theta^2} + \frac{2c_2}{\theta^3}\right)e^{\theta\mu_2} + \left(\frac{2c_1}{\theta^3} - \frac{b_1 + 2c_1\mu_1}{\theta^2}\right)e^{\theta\mu_1} \tag{47}$$

The total amount of items which is perish within the time interval $[0, t_1]$ is

$$D_T = S - \int_0^{t_1} R(t)dt = S - \int_0^{\mu_1}(a_1 + b_1 t + c_1 t^2)dt - \int_{\mu_1}^{\mu_2} R_0 dt - \int_{\mu_2}^{t_1}(a_2 + b_2 t - c_2 t^2)dt$$

$$-\frac{a_1}{\theta} + \frac{b_1}{\theta^2} - \frac{2c_1}{\theta^3} + \left(\frac{a_2 + b_2 t_1 - c_2 t^2}{\theta} - \frac{b_2 - 2c_2 t_1}{\theta^2} - \frac{2c_2}{\theta^3}\right)e^{\theta t_1}$$

$$+ \left(-\frac{b_1 + 2c_1\mu_1}{\theta^2} + \frac{2c_1}{\theta^3}\right)e^{\theta\mu_1} + \left(\frac{b_2 - 2c_2\mu_2}{\theta^2} + \frac{2c_2}{\theta^3}\right)e^{\theta\mu_2} - a_1\mu_1 - \frac{b_1}{2}\mu_1^2 - \frac{c_1}{3}\mu_1^3 \qquad (48)$$

$$- R_0(\mu_2 - \mu_1) - a_2(t_1 - \mu_2) - \frac{b_2}{2}(t_1^2 - \mu_2^2) + \frac{c_2}{3}(t_1^3 - \mu_2^3)$$

The total amount of inventory carried during the time interval $[0, t_1]$ is

$$C_T = \int_0^{t_1} I(t)dt$$

$$= \int_0^{\mu_1} I(t)dt + \int_{\mu_1}^{\mu_2} I(t)dt + \int_{\mu_2}^{t_1} I(t)dt$$

$$= \int_0^{\mu_1}\left[\begin{array}{l}\frac{b_1 + 2c_1 t}{\theta^2} - \frac{2c_1}{\theta^3} - \frac{a_1 + b_1 t + c_1 t^2}{\theta} + \left(\frac{a_2 + b_2 t_1 - c_2 t_1^2}{\theta} - \frac{b_2}{\theta^2} + \frac{2c_2 t_1}{\theta^2} - \frac{2c_2}{\theta^3}\right)e^{\theta(t_1 - t)} \\ + \left(\frac{b_2 - 2c_2\mu_2}{\theta^2} + \frac{2c_2}{\theta^3}\right)e^{\theta(\mu_2 - t)} + \left(\frac{2c_1}{\theta^3} - \frac{b_1 + 2c_1\mu_1}{\theta^2}\right)e^{\theta(\mu_1 - t)}\end{array}\right]dt$$

$$+ \int_{\mu_1}^{\mu_2}\left[\begin{array}{l}-\frac{R_0}{\theta} + \left(\frac{a_2 + b_2 t_1 - c_2 t_1^2}{\theta} - \frac{b_2}{\theta^2} + \frac{2c_2 t_1}{\theta^2} - \frac{2c_2}{\theta^3}\right)e^{\theta(t_1 - t)} \\ + \left(\frac{b_2 - 2c_2\mu_2}{\theta^2} + \frac{2c_2}{\theta^3}\right)e^{\theta(\mu_2 - t)}\end{array}\right]dt \qquad (49)$$

$$+ \int_{\mu_2}^{t_1}\left[\left(\frac{a_2 + b_2 t_1 - c_2 t_1^2}{\theta} - \frac{b_2}{\theta^2} + \frac{2c_2 t_1}{\theta^2} - \frac{2c_2}{\theta^3}\right)e^{\theta(t_1 - t)} + \frac{b_2 - 2c_2 t}{\theta^2} + \frac{2c_2}{\theta^3} - \frac{a_2 + b_2 t - c_2 t^2}{\theta}\right]dt$$

$$= \left(\frac{a_2 + b_2 t_1 - c_2 t_1^2}{\theta} + \frac{2c_2 t_1 - b_2}{\theta^2} - \frac{2c_2}{\theta^3}\right)\left(\frac{e^{\theta t_1} - 1}{\theta}\right) + \left(\frac{b_2 - 2c_2\mu_2}{\theta^2} - \frac{2c_2}{\theta^3}\right)\left(\frac{e^{\theta\mu_2} - 1}{\theta}\right)$$

$$+ \left(\frac{2c_1}{\theta^3} - \frac{b_1 + 2c_1\mu_1}{\theta^2}\right)\left(\frac{e^{\theta\mu_1} - 1}{\theta}\right) - \frac{a_1\mu_1}{\theta} - \frac{b_1\mu_1^2}{2\theta} - \frac{c_1\mu_1^3}{3\theta} + \left(\frac{b_1}{\theta^2} - \frac{2c_1}{\theta^3}\right)\mu_1 + \frac{c_1\mu_1^2}{\theta^2}$$

$$- \frac{R_0}{\theta}(\mu_2 - \mu_1) - \frac{a_2}{\theta}(t_1 - \mu_2) - \frac{b_2}{2\theta}(t_1^2 - \mu_2^2) + \frac{c_2}{3\theta}(t_1^3 - \mu_2^3)$$

$$+ \left(\frac{b_2}{\theta^2} - \frac{2c_2}{\theta^3}\right)(t_1 - \mu_2) - \frac{c_2}{\theta^2}(t_1^2 - \mu_2^2)$$

Total quantity of shortage during the time interval $[t_1, T]$ is

$$B_T = -\int_{t_1}^T I(t)dt = -\int_{t_1}^T\left[a_2(t_1 - t) + \frac{b_2}{2}(t_1^2 - t^2) + \frac{c_2}{3}(t^3 - t_1^3)\right]dt$$

$$= -a_2 t_1(T - t_1) + \frac{a_2}{2}(T^2 - t_1^2) - \frac{b_2}{2}t_1^2(T - t_1) + \frac{b_2}{6}(T^3 - t_1^3) - \frac{c_2}{12}(T^4 - t_1^4) + \frac{c_2 t_1^3}{3}(T - t_1) \qquad (50)$$

Then, the total average cost per unit time under the time interval $\mu_2 \le t_1 \le T$, can be written as

$$A_3(t_1) = \frac{1}{T}[k_0 + k_1 D_T + k_2 C_T + k_3 B_T] \tag{51}$$

The first order derivative of $A_3(t_1)$ with respect to t_1 is as follows:

$$\frac{dA_3(t_1)}{dt_1} = \frac{1}{T}\left[\left(k_1 + \frac{k_2}{\theta}\right)(e^{\theta t_1} - 1) + k_3(t_1 - T)\right](a_2 + b_2 t_1 - c_2 t_1^{\,2}) \tag{52}$$

The required necessary condition for $A_3(t_1)$ to be minimized is

$\frac{dA_3(t_1)}{dt_1} = 0$, that is

$$\frac{1}{T}\left[\left(k_1 + \frac{k_2}{\theta}\right)(e^{\theta t_1} - 1) + k_3(t_1 - T)\right](a_1 + b_1 t_1 + c_1 t_1^{\,2}) = 0 \tag{53}$$

This implies that

$$\left[\left(k_1 + \frac{k_2}{\theta}\right)(e^{\theta t_1} - 1) + k_3(t_1 - T)\right] = 0 \tag{54}$$

Let $p(t_1) = \left[\left(k_1 + \frac{k_2}{\theta}\right)(e^{\theta t_1} - 1) + k_3(t_1 - T)\right]$, $\tag{55}$

since $p'(t_1) = \left(k_1 + \frac{k_1}{\theta}\right)e^{\theta t_1}\theta + k_3 > 0$, which implies that $p(t_1)$ is strictly monotonically increasing function within the interval $t_1 \in [\mu_2, T]$.

Property-3

In this case, the inventory model under the condition $\mu_2 \leq t_1 < T$, $A_3(t_1)$ attains its minimum at $t_1 = t_1^*$, where $p(t_1^*) = 0$ if $\mu_2 < t_1^*$. On the other hand, $A_3(t_1)$ attains its minimum at $t_1^* = \mu_2$ if $t_1^* < \mu_2$. Now, we can calculate the total back-order quantity at the end of the cycle is

$$\Delta_3 = a_2(T - t_1^*) + \frac{b_2}{2}(T^2 - t_1^{*2}) + \frac{c_2}{3}(t_1^{*3} - T^3) \ .$$

Therefore, the optimal order quantity, denoted by Q^*, is $Q^* = S^* + \Delta_3$, where S^* denotes the optimal value of S. From the above three cases, we can derive the following results

Result-1

An inventory model having constant deteriorating rate with quadratic trapezoidal type demand, the optimal replenishment time is t_1^* and $A_1(t_1)$ attains its minimum at $t_1 = t_1^*$ if and only if $t_1^* < \mu_1$. On the other hand, $A_2(t_1)$ attains its minimum at $t_1 = t_1^*$ if and only if $\mu_1 < t_1^* < \mu_2$ and $A_3(t_1)$ attains its minimum at $t_1 = t_1^*$ if and only if $\mu_2 < t_1^*$, where t_1^* is the unique solution of equation $p(t_1) = 0$.

Example

We can consider suitable values of the following parameters as follows: $T = 12$ weeks, $\mu_1 = 4$ weeks, $\mu_2 = 10$ weeks, $a_1 = 100$ unit, $b_1 = 5$ unit, $c_1 = 4$ unit, $a_2 = 220$ unit, $b_2 = 10$ unit, $c_2 = 2$ unit, $\theta = 0.1$, $k_0 = \$200$, $k_1 = \$3$ per unit, $k_2 = \$10$ per unit, $k_3 = \$4$ per unit. Using the above data, we can find $p(\mu_1) = 98.0951 > 0$,

the optimal replenishment time t_1^*=4.397 weeks, the optimal order quantity Q^*, for each ordering cycle, is 4422.3465 unit and the minimum cost $A_1(t_1^*)$=\$5848.1098

Table 1
Sensitivity analysis

Paramete	(%)	t_1^*	Q^*	$A_1(t_1^*)$	Paramete	(%)	t_1^*	Q^*	$A_1(t_1^*)$
a_1	+50	4.82	4688.507	5546.2	c_2	+50	4.637	4547.5	5753.0
	+25	4.71	4616.024	5538.8		+25	4.592	4498.3	5742.4
	+20	4.67	4582.499	5536.2		+20	4.581	4485.6	5740.2
	+10	4.61	4536.084	5534.3		+10	4.558	4459.2	5735.8
	-10	4.43	4448.777	5527.3		-10	4.522	4426.4	5725.7
	-20	4.28	4188.366	5523.2		-20	4.486	4403.5	5722.1
	-25	4.13	4117.554	5514.7		-25	4.471	4391.4	5719.6
	-50	3.99	3968.533	5541.4		-50	4.393	4333.1	5707.0
b_1	+50	4.53	4549.407	5646.0	Θ	+50	4.748	4573.6	5865.2
	+25	4.5	4523.039	5634.9		+25	4.688	4495.6	5857.8
	+20	4.47	4474.111	5530.1		+20	4.566	4473.4	5851.3
	+10	4.42	4436.456	5525.3		+10	4.51	4422.3	5848.1
	-10	4.3	440.8903	5519.7		-10	4.487	4379.8	5841.6
	-20	4.28	4385.378	5510.3		-20	4.417	4366.9	5835.3
	-25	4.21	4325.333	5499.8		-25	4.34	4359.8	5830.6
	-50	4.18	4300.178	5487.3		-50	4.319	4344.5	5823.1
c_1	+50	4.05	4665.328	5230.3	k_1	+50	3.57	3725.3	4955.6
	+25	4.29	4223.772	4886.8		+25	4.076	4057.7	4703.2
	+20	4.33	3943.030	4815.8		+20	4.294	4152.7	4677.2
	+10	4.42	3689.353	4675.3		+10	4.375	4326.9	4605.4
	-10	4.74	3187.698	4387.0		-10	4.552	4560.3	4465.4
	-20	4.81	2931.327	4342.3		-20	4.664	4648.0	4397.6
	-25	5.27	2693.037	4367.8		-25	4.854	4702.0	4302.0
	-50	5.42	2114.598	4295.3		-50	4.908	4798.8	4195.0
a_2	+50	4.94	4877.508	5916.9	k_2	+50	3.927	3744.7	4675.3
	+25	4.91	4803.015	5876.1		+25	3.832	3678.8	4614.2
	+20	4.88	4771.499	5842.5		+20	3.723	3635.4	4600.7
	+10	4.81	4726.773	5811.2		+10	3.687	3548.8	4569.2
	-10	4.76	4710.000	5786.9		-10	3.613	3333.0	4485.1
	-20	4.68	4694.886	5774.5		-20	3.456	3186.2	4427.3
	-25	4.53	4617.663	5737.7		-25	3.341	3106.5	4392.8
	-50	4.47	4579.003	5689.1		-50	3.28	2537.8	4113.9
b_2	50	3.88	3778.544	5745.6	k_3	50	4.267	3823.4	4437.6
	25	4.38	4230.436	5682.3		25	3.883	3752.9	4485.3
	20	4.44	4278.083	5622.8		20	3.826	3658.2	4492.0
	10	4.54	4375.889	5577.7		10	3.652	3630.9	4512.7
	-10	4.62	4530.839	5484.8		-10	3.056	3577.4	4544.3
	-20	4.69	4580.458	5437.9		-20	2.866	3109.2	4564.3
	-25	4.73	4625.320	5412.2		-25	2.544	3014.6	4604.9
	-50	4.99	4673.683	5392.3		-50	1.987	2995.1	4687.3

In the above table some sensitivity analysis of the model is performed by changing the parameter -50%, -25%, -20%, -10%, +10%, +20%, +25% and +50% taking one at a time and keeping the remaining unchanged.

4. Conclusion

In a realistic product life cycle, demand is increasing with time during the growth phase. Then, after reaching its peak, the demand becomes stable for a finite time period called the maturity phase. Thereafter, the demand starts decreasing with time. Therefore, in this paper, we have studied the inventory model for constant deteriorating items with quadratic trapezoidal demand rate. We have proposed an inventory replenishment policy for this type of inventory model. From the market information, we have found that the quadratic trapezoidal type demand rate was more realistic than ramp type demand rate, constant demand rate and other time dependent demand rate Our paper provides an interesting topic for the future study of such kind of important inventory models, and at the same time, the following problems can be considered for future research work (1) How about the inventory model starting with shortages? (2) How about the inventory model with time dependent deteriorating rate instead of constant deteriorating rate?

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