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Periodic inventory model with controllable lead time where backorder rate depends on protection interval

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A B S T R A C T

In this paper, a period review inventory model with controllable lead time has been considered where shortages are partially backlogged. The backorder rate is dependent on the backorder discount and the length of the protection interval, which is sum of the review period and the lead time. Two cases have been discussed for protection interval demand which are (a) Demand distribution is known (Normal Distribution) (b) Demand distribution is unknown (Minimax distribution). Further, algorithms have been developed which jointly optimize the backorder discount, the review period and the lead time for each case. Numerical examples are also presented to illustrate the results.

1. Introduction

In the recent inventory control system, modern enterprises realize the importance of managing inventory efficiently to run the system profitably. A renowned Just-in-time (JIT) philosophy emphasizes on the advantages and benefits associated with reducing the lead time. Lead time is a topic of interest in most of the inventory systems. Generally, it is assumed that lead time is prescribed (Deterministic and Stochastic) and which therefore is not subject to control. Tersine (1982) suggested that order preparation, order transit supplier lead time, delivery time and setup time (i.e. preparation time for availability of items) usually constitute the total lead time of the system. The lead-time can be decomposed into several crashing periods for making the present system more effective. In many practical situations, lead time can be reduced at an added crashing cost; in other words it is controllable. By shortening the lead time, we can lower the safety stock, reduce the loss caused by stock out; improve the service level to the customer and increase the competitive ability in business. Many researchers Liao and Shyu (1991), Moon and Choi (1998), Hariga and Ben-Daya (1999) have investigated continuous review inventory models with lead time as a decision variable.

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It is generally observed that while shortage occurs, demand can be captured partially. Some customers may prefer their demands to be backordered i.e., some customers whose needs are not urgent can wait for their demands to be satisfied, while others who cannot wait have to fill their demand from another source which is lost sale case. However, certain factors motivate the customer for the backorders out of which price discount from the supplier is the crucial one. To some extent, sufficient price discounts to the customers help the supplier to secure more backorders through negotiation. The supplier could fetch a large number of back order rate with higher price discount. Pan and Hsiao (2001) presented continuous inventory model with backorder discounts and variable lead-time. In this paper, the backorder discount has been also taken as a one of the decision variables. Further, the backorder rate depends on the length of the protection interval (period during which shortages can occur) also. This fact point out that when shortages occur, if longer the length of protection interval is, then, larger the amount of shortages is and obviously, this results in smaller the proportion of customers who can wait their orders to be fulfilled and results in smaller backorder rate. The consideration is 'unsatisfied demand during the shortages can lead to optimal backorder ratio by controlling the price discount and the length of protection interval' which ultimately helps the supplier to minimize his total inventory cost.

In a recent study, Pan and Hsiao (2005) expanded the continuous inventory model by considering the case where lead-time crashing cost is taken as the function of reduced lead-time and ordered quantities. In contrast to the continuous review inventory model, we seek to investigate a periodic review model with back order discounts to accommodate more practical feature of the tangible Inventory systems. The applications of periodic review inventory model can often be found in managing inventory cases such as smaller retailer stores, drugs stores and grocery stores by Taylor (1996). Earlier, Chuang et al. (2004) presented a periodic review inventory model with variable lead-time and reduction of setup cost. Jaggi and Arneja (2010) considered a periodic inventory model with unstable lead-time and setup cost with backorder rate depending on backorder discount only. The main objective of this study is to uncover the benefits associated with reduction of lead time and offering backorder discount where backorder rate is dependent on length of protection interval. Two cases have been discussed for protection interval demand

1. Distribution is known
2. Demand distribution is unknown

In this study, an inventory model has been formulated which allows review period, lead time and backorder discount to be optimized with known service level. The lead-time is also controllable and has shown that the significant saving could be obtained by offering suitable backorder discount.

2. Notations and assumptions

To develop the proposed model, we have used the following notation and assumptions:

2.1 Notations

D **:** Average demand per year

K : Fixed ordering cost per inventory cycle

h : Inventory holding cost per unit per year

R : Target Level

β : Fraction of the demand back ordered during stock out period such as $0 \le \beta \le 1$

β_0 : Upper bound of the backorder rate

π_0 : Marginal Profit (i.e. cost of lost demand) per unit

π_x : Back order price discount offered by the supplier per unit

L : Length of lead-Time

X : Protection interval demand which has a p. d. f. f_x with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$ (>0) for the protection interval ($T+L$) where σ denotes the standard deviation of the demand per unit.

Ω : The class of p. d. f. f_x of the protection interval demand with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$

S : Fixed shortages cost, \$ per unit short

A : Safety factor

T : Length of a review period

$E(.)$: Mathematical Expectation

X^+ : Maximum value of x and 0 i.e., $X^+ = \max\{x,0\}$

EAC : Expected annual cost

EAC^W : Least upper bound of expected annual cost.

2.2 *Assumptions*

1. The inventory level is reviewed every T units of time. A sufficient quantity is ordered up to the target level R, and the ordering quantity is arrived after L units of time.
2. The length of the lead-time L does not exceed an inventory cycle time T so that there is never more than a single order outstanding in any cycle.
3. The target level R = Expected demand during the protection interval + safety stock (SS) i.e. $R = D(T+L) + A\sigma\sqrt{(T+L)}$ where A is the safety factor and satisfies $P[x > R] = q$, q represent the allowable stock out probability which means service level is defined during the protection interval and is given.
4. The lead-time L consists of n mutually independent components. The i[th] component has a minimum duration a_i and normal duration b_i, and a crashing cost per unit time c_i. Arranging c_i such that $c_1 \le c_2 \le c_3 \le ... \le c_n$ for the convenience. Since it is clear that the reduction of lead-time should be first on component 1 because it has the minimum unit crashing cost, and then component 2, and so on.
5. Let $L_0 = \sum_{j=1}^{n} b_j$ and L_i be the length of lead time with components 1,2,…,i crashed to their minimum duration, then L_i can be expressed as $L_i = L_0 - \sum_{j=1}^{i}(b_j - a_j)$, , i= 1,…,n and the lead time crashing cost per cycle $C(L)$ is given as $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$. (Ouyang et al., 1996).
6. Assuming that a fraction $\beta\,(0 \le \beta \le 1)$ of the demand during the stock out period can be backordered so the remaining fraction $1-\beta$ is lost. The backorder rate β is variable and is in proportion to the price discount π_x offered by the supplier per unit and the protection interval. Thus $\beta = \dfrac{T\beta_0\pi_x}{(T+L)\pi_0}$ where $0 \le \beta_0 \le 1$ and $0 \le \pi_x \le \pi_0$, here our model is different from the previous models. (Pan & Hsiao, 2005).

3. Mathematical Model

We have assumed that the protection interval demand X has a p. d. f. f_x with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$ with the target level $R = D(T+L) + A\sigma\sqrt{(T+L)}$ where A is already defined.

As Ouyang and Chaung (1999) proposed the periodic review model where the expected net inventory at the beginning of the period is $R-DL+(1-\beta)E(X-R)^{+}$.

Therefore, the expected net inventory at the end of the period is $R-DL-DT+(1-\beta)E(X-R)^{+}$ which gives the expected holding cost per year approximately $h\left[R-DL-\frac{DT}{2}+(1-\beta)E(X-R)^{+}\right]$.

Now, the expected stock out cost per year is $\frac{\pi_x \beta E(X-R)^{+}+(S+\pi_0)(1-\beta)E(X-R)^{+}}{T}$ where $E(X-R)^{+}$ is the expected demand shortage at the end of cycle i.e., $E(X-R)^{+}=\int_R^{\infty}(x-R)f_x dx$.

When the lead time L is reduced to L_i then, Annual lead time crashing cost $=\frac{\sum_{j=1}^{i} c_j}{T}$

Now the objective is to minimize the total expected annual cost (EAC) which is the sum of
= Ordering cost + Stockout cost + Holding cost + Lead-time crashing cost

$$EAC(T,\pi_x,L)=\frac{K}{T}+\frac{\pi_x \beta E(X-R)^{+}+(S+\pi_0)(1-\beta)E(X-R)^{+}}{T}+h\left[R-DL-\frac{DT}{2}+(1-\beta)E(X-R)^{+}\right]+\frac{\sum_{j=1}^{i} c_j}{T} \quad (1)$$

Also, we have assumed that the backorder rate β depends on the backorder price discount π_x and protection interval $(T+L)$. Thus $\beta=\frac{T\beta_0 \pi_x}{(T+L)\pi_0}$ and $R=D(T+L)+A\sigma\sqrt{(T+L)}$, where A is safety factor.

The Eq. (1) can be written as

$$EAC(T,\pi_x,L)=\frac{K+\sum_{j=1}^{i} c_j}{T}+\frac{E(X-R)^{+}\left[\frac{T\beta_0 \pi_x^2}{(T+L)\pi_0}+S+\pi_0-\frac{ST\beta_0 \pi_x}{(T+L)\pi_0}-\frac{T\beta_0 \pi_x}{(T+L)}\right]}{T}$$
$$+h\left[\frac{DT}{2}+A\sigma\sqrt{T+L}+\left(1-\frac{T\beta_0 \pi_x}{(T+L)\pi_0}\right)E(X-R)^{+}\right] \quad (2)$$

Here two cases arise for distribution of lead time demand i.e.

a. Normal distribution
b. Unknown distribution

3.1 Lead time demand with normal distribution

In this section, we have assumed that the probability distribution of protection interval demand X has a normal distribution with mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$

So, the expected shortages occurring at the end of the cycle is given by

$$E(X-R)^{+}=\int_R^{\infty}(x-R)f_x dx=\sigma\sqrt{T+L}\,\psi(A)>0$$

Where $\psi(A)=\varphi(A)-A\left[1-\Phi(A)\right]$, φ and Φ are the standard normal p. d. f. and c. d. f., respectively.

Therefore, Eq. (2) is reduced to

$$EAC(T,\pi_x,L)=\frac{K+\sum_{j=1}^{i} c_j}{T}+\frac{\sigma\sqrt{T+L}\Psi(A)\left[\frac{T\beta_0 \pi_x^2}{(T+L)\pi_0}+S+\pi_0-\frac{ST\beta_0 \pi_x}{(T+L)\pi_0}-\frac{T\beta_0 \pi_x}{(T+L)}\right]}{T}+h\left[\frac{DT}{2}+A\sigma\sqrt{T+L}+\left(1-\frac{T\beta_0 \pi_x}{(T+L)\pi_0}\right)\sigma\sqrt{T+L}\Psi(A)\right] \quad (3)$$

It can be checked that for fixed T and π_x, $EAC(T,\pi_x,L)$ is a concave function of $L \in \left(L_i, L_{i-1}\right)$ because $\frac{\partial^2 EAC(T,\pi_x,L)}{\partial L^2} < 0$. So, for fixed (T,π_x,L), the minimum total expected annual cost will occur at the end points of the interval $\left(L_i, L_{i-1}\right)$. On the other hand, for a given value of $L \in \left(L_i, L_{i-1}\right)$, it can be shown that $EAC(T,\pi_x,L)$ is convex in (T,π_x). Thus for fixed $L \in \left(L_i, L_{i-1}\right)$, the minimum value of $EAC(T,\pi_x,L)$ will occur at the point (T,π_x) that satisfy $\frac{\partial EAC(T,\pi_x,L)}{\partial T} = 0$ and $\frac{\partial EAC(T,\pi_x,L)}{\partial \pi_x} = 0$. Now, $\frac{\partial EAC(T,\pi_x,L)}{\partial T} = 0 \Rightarrow$

$$-\frac{K+\sum_{j=1}^{i} c_j}{T^2} + \sigma\psi(A)\left[\begin{array}{l} -\frac{1}{2}\frac{\beta_0\pi_x^2 (T+L)^{-\frac{3}{2}}}{\pi_0} + \left(S+\pi_0\right)\left\{-\frac{(T+L)^{\frac{1}{2}}}{T^2} + \frac{(T+L)^{-\frac{1}{2}}}{2T}\right\} \\ + \frac{S\beta_0\pi_x (T+L))^{-\frac{3}{2}}}{2\pi_0} + \frac{\beta_0\pi_x (T+L))^{-\frac{3}{2}}}{2} \end{array}\right] + h\left[\begin{array}{l} \frac{D}{2} + \frac{A\sigma}{2}(T+L)^{-\frac{1}{2}} + \frac{\sigma\Psi(A)(T+L)^{-\frac{1}{2}}}{2} \\ + \frac{T\beta_0\pi_x\sigma\Psi(A)(T+L)^{-\frac{3}{2}}}{2\pi_0} \end{array}\right] \tag{4}$$

This can be written as

$$\frac{(K+\sum_{j=1}^{i} c_j)}{T^2} + \sigma\psi(A)\left[\begin{array}{l} \frac{1}{2}\frac{\beta_0\pi_x^2 (T+L)^{-\frac{3}{2}}}{\pi_0} + \left(S+\pi_0\right)\left\{\frac{(T+L)^{\frac{1}{2}}}{T^2} - \frac{(T+L)^{-\frac{1}{2}}}{2T}\right\} \\ - \frac{S\beta_0\pi_x (T+L))^{-\frac{3}{2}}}{2\pi_0} - \frac{\beta_0\pi_x (T+L))^{-\frac{3}{2}}}{2} \end{array}\right] = h\left[\begin{array}{l} \frac{D}{2} + \frac{A\sigma}{2}(T+L)^{-\frac{1}{2}} + \frac{\sigma\Psi(A)(T+L)^{-\frac{1}{2}}}{2} \\ + \frac{T\beta_0\pi_x\sigma\Psi(A)(T+L)^{-\frac{3}{2}}}{2\pi_0} \end{array}\right] \tag{5}$$

where

$$\frac{\partial EAC(T,\pi_x,L)}{\partial \pi_x} = 0 \Rightarrow \pi_x = \frac{(S+\pi_0)+hT}{2} \tag{6}$$

Since it is difficult to obtain the solution for T and π_x explicitly as the evolution of Eq. (5) and Eq. (6) need the value of each other. As a result, we must establish the following iterative algorithm to find the optimal (T,π_x).

Algorithm 3.1.1

Step 1 For each $L_i, i = 0,1,2,...,n$, execute (a) – (b)

(a) Substitute the value of $\psi(A_i)$ into Eq. (5), using numerical search technique, evaluate T_i. If $T_i \geq L_i$, then go to (b) otherwise let $T_i = L_i$, go to (b).

(b) Substitute the value of T_i, in Eq. (6) to obtain the value of π_{x_i}. Compare π_{x_i} and π_0.

If $\pi_{x_i} \leq \pi_0$, then π_{x_i} is feasible. Go to step (2) otherwise let $\pi_{x_i} = \pi_0$, go to step (2).

Step 2 For each (T_i, π_{x_i}, L_i), Compute the corresponding expected annual cost EAC (T_i, π_{x_i}, L_i), from Eq. (3). Go to step 3.

Step 3 Find $\min_{i=0,1,2,...,n} EAC\ (T_i, \pi_{x_i}, L_i)$.

Let $EAC\ (T^*, \pi_x^*, L^*) = \min_{i=0,1,2,...,n} EAC\ (T_i, \pi_{x_i}, L_i)$, Hence ($T^{*}, \pi_x^{*}, L^{*}$) is the optimal solution and the optimal target level is $R^{*} = D(T^{*}+L^{*}) + A\sigma\sqrt{T^{*}+L^{*}}$. Theoretically, for given K, D, h, β_0, π_0, σ and each L_i (i = 0, 1, 2, ... n), from Eq. (5) and Eq. (6), we can obtain optimal values of T and π_x, then the corresponding total expected annual cost can be found. Thus, the minimum total expected annual cost could be obtained when the lead–time demand is normally distributed.

3.2 Lead time demand with unknown distribution

If the lead time demand does not follow normal distribution or the probability distribution is unknown with first two moments, then the solution can be obtained by minimax approach. Since the probability distribution of X is unknown, we cannot find the exact value of $E(X-R)^{+}$. Now we use a minimax distribution free procedure to solve $\min_{T>0,\pi_x>0,L>0} \max_{F\in\Omega} EAC(T,\pi_x,L)$, we need the following proposition to shorten the problem.

Proposition 3.2.1

For any $F \in \Omega$,

$$E(X-R)^{+} \le \frac{1}{2}\left(\begin{array}{l}\sqrt{\sigma^2(T+L)+(R-D(T+L))^2} \\ -\big(R-D(T+L)\big)\end{array}\right) \le \frac{1}{2}\,\sigma\sqrt{(T+L)}\left(\sqrt{1+A^2}-A\right) \tag{7}$$

(Chuang et al.(2004))

Moreover, the upper bound (7) is tight. Then the Eq. (2) can be reduced to

$$EAC^{W}(T,\pi_x,L) = \frac{K+\sum_{j=1}^{i} c_j}{T} + \frac{\sigma\times\sqrt{T+L}\times\frac{1}{2}\times(\sqrt{1+A^2}-A)\left[\begin{array}{l}\dfrac{T\beta_0\pi_x^2}{(T+L)\pi_0}+S+\pi_0 \\ -\dfrac{ST\beta_0\pi_x}{(T+L)\pi_0}-\dfrac{T\beta_0\pi_x}{(T+L)}\end{array}\right]}{T}$$
$$+h\left[\frac{DT}{2}+A\sigma\sqrt{T+L}+\left(1-\frac{T\beta_0\pi_x}{(T+L)\pi_0}\right)\times\sigma\sqrt{T+L}\,\frac{1}{2}\times(\sqrt{1+A^2}-A)\right] \tag{8}$$

where $EAC^{W}(T,\pi_x,L)$ is the least upper bound of $EAC(T,\pi_x,L)$. As notified in the preceding section, it can be shown that $EAC^{W}(T,\pi_x,L)$ is a concave function of $L\in\left(L_i,L_{i-1}\right)$ for fixed T and π_x [Appendix A].Therefore, the minimum upper bound of the expected total annual cost will occur at the end point of the interval $L\in\left(L_i,L_{i-1}\right)$ for fixed value of (T, π_x). Moreover, it can be shown that $EAC^{W}(T,\pi_x,L)$ is convex function of T and π_x for fixed L [Appendix B]. Therefore, the first order conditions are necessary and sufficient conditions for optimality. Using the first condition of derivatives, we get

$$\frac{(K+\sum_{j=1}^{i} c_j)}{T^2}+\sigma\psi(A)\left[\begin{array}{l}\dfrac{1}{2}\dfrac{\beta_0\pi_x^2(T+L)^{-\frac{3}{2}}}{\pi_0}+\left(S+\pi_0\right)\left\{\dfrac{(T+L)^{\frac{1}{2}}}{T^2}-\dfrac{(T+L)^{-\frac{1}{2}}}{2T}\right\} \\ -\dfrac{S\beta_0\pi_x(T+L)^{-\frac{3}{2}}}{2\pi_0}-\dfrac{\beta_0\pi_x(T+L)^{-\frac{3}{2}}}{2}\end{array}\right] = h\left[\begin{array}{l}\dfrac{D}{2}+\dfrac{A\sigma}{2}(T+L)^{-\frac{1}{2}}+\dfrac{\sigma\Psi(A)(T+L)^{-\frac{1}{2}}}{2} \\ +\dfrac{T\beta_0\pi_x\sigma\Psi(A)(T+L)^{-\frac{3}{2}}}{2\pi_0}\end{array}\right] \tag{9}$$

and

$$\pi_x = \frac{(S+\pi_0)+hT}{2} \tag{10}$$

Since it is difficult to obtain the exact value of service factor A which depends upon the required service level on the basis of allowable stock out probability q, because the p. d. f. $f_X(x)$ is unknown. So, the following proposition has been used to find accurate value of A. Therefore, the algorithm to find the optimal review period, lead-time and backorder discount can be established by using the proposition given below:

Proposition 3.2.2

Let X represents the protection interval demand that has p. d. f. $f_X(x)$ with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$ then for any real number c > 0, $p\left[X > c\right] \le \dfrac{\sigma^2 L}{\sigma^2 L + (c-DL)^2}$. If we take R instead of c, then $p\left[X > R\right] \le \dfrac{\sigma^2 (T+L)}{\sigma^2 (T+L) + (R-DL)^2}$, which implies $p\left[X > R\right] \le \dfrac{1}{1+A^2}, q = p\left[X > R\right] \le \dfrac{1}{1+A^2}$

Hence, $q = \dfrac{1}{1+A^2}$, Therefore $A \in \left[0, \sqrt{\dfrac{1}{q}-1}\right]$ (Chuang et al. (2004))

Algorithm 3.2.3

Step 1 For each q, divide the interval $\left[0, \sqrt{\dfrac{1}{q}-1}\right]$ into N equal subintervals. Let $A_0 = 0$, $A_N = \sqrt{\dfrac{1}{q}-1}$

$$A_l = A_{l=1} + \frac{A_N - A_0}{N}, \quad l = 1,2,...,N-1$$

Step 2 For each L_i $(i = 0,1,2,...,n)$, perform step (3) and (4).

Step 3 For given $A_l \in \left\{A_0, A_1, ..., A_N\right\}$, $l = 0,1,2,...,N$, using numerical search technique, evaluate T_i from Eq. (9) simultaneously.
If $T_i \ge L_i$, then go to step (4) otherwise
Set $T_i = L_i$, and go to step (4).

Step 4 By using T, Calculate the value of π_i using the Eq. (10). Compare π_i and π_0.
If $\pi_i \le \pi_0$. Then π_i is feasible. Go to next step. Otherwise set
$\pi_i = \pi_0$. Go to step (5).

Step 5 For each (T_i, π_{x_i}, L_i), Compute the corresponding expected annual cost EAC^W (T_i, π_{x_i}, L_i).

Step 6 Find $\min\limits_{A_l \in \left\{A_0, A_1, ..., A_N\right\}} EAC^W (T_i, \pi_{x_i}, L_i)$,

Let $EAC^W (T_{i,A_i}, \pi_{x_i,A_i}, L_{i,A_i}) = \min\limits_{A_l \in \left\{A_0, A_1, ..., A_N\right\}} EAC^W (T_i, \pi_{x_i}, L_i)$,

Step 7 Find $EAC^W (T', \pi_x', L') = \min\limits_{i=0,1,2,...,n} EAC^W (T_{i,A_i}, \pi_{x_i,A_i}, L_{i,A_i})$

Then (T', π_x', L') is the required optimal solution.

Numerical Example

In order to illustrate the solution algorithms, we have considered an inventory system with the following data having data: D=600 units per year, K= $ 200 per order, S= $ 50 per short out, $\sigma = 7$ units per week, π_0 = $ 150 per unit, h= $ 20 per unit per year, q = 0.2 where A_0= 0 and A_N = 2, N=200. We have started with fixed service level A = 0.8 (i.e. A_i = 0.845 and $\psi(A_i)$ =0.1120) by checking the table for Silver and Peterson (1985) (p.p. 699-708). The lead-time has three components, which have been shown in Table 1.

Table 1
Lead time data

Lead time component i	Normal duration (days) b_i	Minimum duration (days) a_i	Unit Crashing cost per day, c_i
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

We have solved the cases for different upper bounds of the backorder ration β= 0, 0.5, 0.8, 1. Now, $\beta = 0$, represent complete lost sales; $\beta = 1$, represent complete backorder case and $0 < \beta < 1$ represent partially backorder case. Then applying the algorithm 1, crashing has been carried out for lead-time for different backorder ratio and illustrated in Table 2. It is observed that by reducing the lead time the total expected cost decreases.

Table 2
Crashing (Normal) of lead time when the protection interval demand is known

	L	*T*	π	*R*	*EAC*
$\beta_0 = 0$	8	13.94	77.68	253.70	3390.87
	6	13.79	77.65	228.82	3340.90
	4	**13.78**	**77.65**	**205.63**	**3323.85**
	3	14.19	77.73	198.79	3411.17
$\beta_0 = 0.5$	8	13.86	77.67	252.78	3370.26
	6	13.68	77.63	227.54	3317.74
	4	**13.61**	**77.62**	**203.63**	**3296.93**
		13.95	77.68	196.06	3381.66
$\beta_0 = 0.8$	8	13.81	77.66	252.23	3357.93
	6	13.61	77.62	226.79	3303.94
	4	**13.51**	**77.60**	**202.48**	**3280.99**
	3	13.82	77.66	194.49	3364.32
$\beta_0 = 1$	8	13.78	77.65	251.88	3349.74
	6	13.57	77.61	226.29	3294.77
	4	**13.44**	**77.58**	**201.72**	**3270.45**
	3	13.73	77.64	193.48	3352.91

Table 3
Optimal Solutions when demand has Normal Distribution

β_0	*L*	*T*	π_x	*R*	*EAC*
0	4	13.78	77.65	205.63	3323.85
0.5	4	13.61	77.62	203.63	3296.93
0.8	4	13.51	77.60	202.48	3280.99
1	4	13.44	77.58	201.72	3270.45

Table 3 provides the solution with crashing of lead time with normal distribution. Here we observed that the total annual expected cost decreases as the backorder ratio increases since supplier can fetch a large number of backorders by offering the price discount with no loss although with less cost. The optimal inventory results with relevant savings where lead-time have been crashed given in table 4. In table 5, algorithm 2 has been applied for crashing of lead-time for different backorder ratio when demand during the protection interval is unknown.

Table 4
Savings (%) Obtained by crashing of lead time with normally distributed demand

β_0	Total Cost without crashing	Total Cost with crashing	Savings (%)
0	3390.87	3323.85	1.97
0.5	3370.26	3296.93	2.17
0.8	3357.93	3280.99	2.29
1	3349.74	3270.45	2.36

Note: saving % = $\left[EAC(T,\pi_x,L)-EAC(T^*,\pi_x^*,L^*\right]/\,EAC(T,\pi_x,L)\times 100\%$

Table 5
Crashing of lead time when the protection interval demand is unknown (Minimax)

	L	T	π	R	EAC
$\beta_0=0$	8	16.15	78.11	279.24	3934.22
	6	15.80	78.04	252.03	3836.07
	4	**15.53**	**77.99**	**225.88**	**3758.37**
	3	15.76	78.03	216.90	3802.65
$\beta_0=0.5$	8	15.95	78.07	276.87	3891.07
	6	15.52	77.98	248.83	3787.61
	4	**15.12**	**77.91**	**221.11**	**3702.00**
	3	15.21	77.93	210.62	3740.84
$\beta_0=0.8$	8	15.83	78.04	275.50	3865.45
	6	15.36	77.95	247.00	3759.03
	4	**14.89**	**77.86**	**218.45**	**3669.23**
	3	14.91	77.87	207.16	3705.50
$\beta_0=1$	8	15.75	78.03	274.61	3848.48
	6	15.26	77.93	245.83	3740.16
	4	**14.74**	**77.84**	**216.75**	**3647.77**
	3	14.72	77.83	204.98	3682.57

Furthermore, Table 6 listed the optimal result for controllable lead-time with unknown distribution.

Table 6
Optimal Solutions when demand has unknown Distribution

β_0	L	T	π	R	EAC
0	4	15.53	77.99	225.88	3758.37
0.5	4	15.12	77.91	221.11	3702.00
0.8	4	14.89	77.86	218.45	3669.23
1	4	14.74	77.84	216.75	3647.77

Table 7
Savings (%) Obtained by crashing of lead time with unknown distribution of demand

β_0	Total Cost without crashing	Total Cost with crashing	Savings (%)
0	3934.22	3758.37	4.46
0.5	3891.07	3702.00	4.85
0.8	3865.45	3669.23	5.07
1	3848.48	3647.77	5.21

4. Conclusion

In the proposed model, the effect of backorder discount and length of protection interval on backorder rate with the reduction of lead time in periodic review model has been considered. Reduction in lead time plays an important role to run the system profitably as it helps the supplier to reduce the overall cost of the system by reducing the loss caused by shortages and improving the service level to the customers. Further, longer length of the protection interval results as large amount of shortages and obviously small proportion of customers who can wait their orders to be fulfilled which means smaller backorder rate. Thus, the reduction of lead time and backorder discount are two significant factors which help the supplier to increase his backorder rate and to earn more profit. This model jointly optimizes the review period, lead time and backorder discount. Further, we consider both cases of protection interval demand with known distribution and unknown distribution.

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Appendix A

Proof of $EAC^{W}(T,\pi_x,L)$ is concave in $L\in\left(L_i,L_{i-1}\right)$ for fixed T,π_x.

$$\frac{\partial EAC^{W}(T,\pi_x,L)}{\partial L}=\frac{1}{T}\left[\sigma\times\Psi(A)\left\{\begin{array}{l}\dfrac{-T\beta_0\pi_x^2(T+L)^{-\frac{3}{2}}}{2\pi_0}+\dfrac{(S+\pi_0)(T+L)^{-\frac{1}{2}}}{2}+\\ \left(\dfrac{ST\beta_0\pi_x}{\pi_0}+T\beta_0\pi_x\right)\dfrac{(T+L)^{-\frac{3}{2}}}{2}\end{array}\right\}\right]$$
$$+\left[\frac{hA\sigma\left(T+L\right)^{-\frac{1}{2}}}{2}+\left(\frac{(T+L)^{-\frac{1}{2}}}{2}+\frac{T\beta_0\pi_x(T+L)^{-\frac{3}{2}}}{2\pi_0}\right)h\,\sigma\,\Psi(A)\right] \quad \text{(A.1)}$$

$$\frac{\partial^2 EAC^{W}(T,\pi_x,L)}{\partial L^2}=\frac{1}{T}\left[\sigma\times\Psi(A)\left\{\begin{array}{l}\dfrac{T\beta_0\pi_x^2(T+L)^{-\frac{5}{2}}}{4\pi_0}-\dfrac{(S+\pi_0)(T+L)^{-\frac{3}{2}}}{4}\\ -\dfrac{3}{4}\left(\dfrac{ST\beta_0\pi_x}{\pi_0}+T\beta_0\pi_x\right)(T+L)^{-\frac{5}{2}}\end{array}\right\}\right]$$
$$-\frac{hA\sigma}{4}(T+L)^{\frac{-3}{2}}-\left[\frac{(T+L)^{-\frac{3}{2}}}{4}+\frac{3T\beta_0\pi_x\times(T+L)^{-\frac{5}{2}}}{4\pi_0}\right]h\sigma\Psi(A)<0 \quad \text{(A.2)}$$

Therefore EAC is concave in $L\in\left(L_i,L_{i-1}\right)$ for fixed (T,π_x).

Appendix B

$$H = \begin{bmatrix} \dfrac{\partial^2 EAC}{\partial T^2} & \dfrac{\partial^2 EAC}{\partial T \partial \pi_x} \\ \dfrac{\partial^2 EAC}{\partial \pi_x \partial T} & \dfrac{\partial^2 EAC}{\partial^2 \pi_x} \end{bmatrix}$$

$$EAC(T, \pi_x, L) = \frac{K + \sum c_j}{T} + \sigma\psi(A) \begin{bmatrix} \dfrac{\beta_0 \pi_x^2}{\pi_0 \sqrt{T+L}} + (S + \pi_0) \dfrac{\sqrt{T+L}}{T} \\ - \dfrac{S\beta_0 \pi_x}{\pi_0 \sqrt{T+L}} - \dfrac{\beta_0 \pi_x}{\sqrt{T+L}} \end{bmatrix}$$

$$+ h \left[\frac{DT}{2} + A\sigma\sqrt{T+L} + \left(1 - \frac{T\beta_0 \pi_x}{\pi_0 (T+L)} \right) \sigma\psi(A)\sqrt{T+L} \right] \tag{B.1}$$

$$\frac{\partial EAC(T, \pi_x, L)}{\partial T} = - \frac{K + \sum_{j=1}^{i} c_j}{T^2} + \sigma\psi(A) \begin{bmatrix} -\dfrac{1}{2} \dfrac{\beta_0 \pi_x^2}{\pi_0 (T+L)^{\frac{3}{2}}} - \dfrac{(S+\pi_0)(T+L)^{\frac{1}{2}}}{T^2} + \dfrac{(S+\pi_0)}{2T\sqrt{T+L}} \\ + \dfrac{S\beta_0 \pi_x}{2(T+L)^{\frac{3}{2}} \pi_0} + \dfrac{\beta_0 \pi_x}{2(T+L)^{\frac{3}{2}}} \end{bmatrix}$$

$$+ \frac{hD}{2} + \frac{hA\sigma}{2(T+L)^{\frac{1}{2}}} + h\sigma\Psi(A) \begin{bmatrix} \dfrac{1}{2(T+L)^{\frac{1}{2}}} - \dfrac{\beta_0 \pi_x}{2(T+L)^{\frac{1}{2}}} \\ + \dfrac{T\beta_0 \pi_x}{2(T+L)^{\frac{3}{2}} \pi_0} \end{bmatrix} \tag{B.2}$$

$$\frac{\partial^2 EAC}{\partial T \partial \pi_x} = \sigma\psi(A) \left[- \frac{\beta_0 \pi_x}{(T+L)^{\frac{3}{2}} \pi_0} + \frac{S\beta_0}{2(T+L)^{\frac{3}{2}} \pi_0} + \frac{\beta_0}{2(T+L)^{\frac{3}{2}}} \right]$$

$$- \frac{h\sigma\psi(A)\beta_0}{(T+L)^{\frac{1}{2}} \pi_0} + \frac{T\beta_0 h\sigma\psi(A)}{2(T+L)^{\frac{3}{2}} \pi_0} \tag{B.3}$$

$$\frac{\partial EAC}{\partial \pi_x} = \sigma\psi(A) \begin{bmatrix} \dfrac{2\beta_0 \pi_x}{\pi_0 \sqrt{T+L}} - \dfrac{S\beta_0}{\pi_0 \sqrt{T+L}} \\ - \dfrac{\beta_0}{\sqrt{T+L}} \end{bmatrix} - \frac{hT\beta_0 \sigma\psi(A)}{\pi_0 \sqrt{T+L}} \tag{B.4}$$

$$\frac{\partial^2 EAC}{\partial \pi_x^2} = \frac{2\sigma\psi(A)\beta_0}{\pi_0 \sqrt{T+L}} \tag{B.5}$$

$$\frac{\partial^2 EAC}{\partial T^2}=\frac{2(k+\sum c_j)}{T^3}+\sigma\psi(A)\left[\begin{array}{l}\frac{3\beta_0\pi_x^2}{4\pi_0(T+L)^{\frac{5}{2}}}+\frac{2(S+\pi_0)}{T^3(T+L)^{\frac{-1}{2}}}-\frac{(S+\pi_0)}{T^2(T+L)^{\frac{1}{2}}}\\-\frac{(S+\pi_0)}{4T(T+L)^{\frac{3}{2}}}-\frac{3\beta_0\pi_x(S+\pi_0)}{4(T+L)^{\frac{5}{2}}\pi_0}\end{array}\right]$$

$$-\frac{hA\sigma}{4(T+L)^{\frac{3}{2}}}+h\sigma\psi(A)\left[\begin{array}{l}\frac{\beta_0\pi_x}{(T+L)^{\frac{3}{2}}\pi_0}-\frac{1}{4(T+L)^{\frac{3}{2}}}\\-\frac{3T\beta_0\pi_x}{4(T+L)^{\frac{5}{2}}\pi_0}\end{array}\right]. \tag{B.6}$$

Using

$$\frac{\partial EAC}{\partial T}=0\Rightarrow$$

$$\frac{2}{T}\frac{\left[k+\sum c_j\right]}{T^2}=\frac{2}{T}\left[\begin{array}{l}\sigma\psi(A)\left(\begin{array}{l}\frac{-\beta_0\pi_x^2}{2(T+L)^{\frac{3}{2}}\pi_0}-\frac{(S+\pi_0)}{T^2(T+L)^{-\frac{1}{2}}}+\frac{(S+\pi_0)}{2T(T+L)^{\frac{1}{2}}}\\+\frac{\beta_0\pi_x(S+\pi_0)}{2(T+L)^{\frac{3}{2}}\pi_0}\end{array}\right)\\+\frac{hD}{2}+\frac{hA\sigma}{2(T+L)^{\frac{1}{2}}}\\+h\sigma\psi(A)\left(\frac{1}{2(T+L)^{\frac{1}{2}}}-\frac{\beta_0\pi_x}{\pi_0(T+L)^{\frac{1}{2}}}+\frac{T\beta_0\pi_x}{2(T+L)^{\frac{3}{2}}\pi_0}\right)\end{array}\right] \tag{B.7}$$

We get

$$\frac{\partial^2 EAC}{\partial T^2}=\sigma\psi(A)\left[\frac{\beta_0\pi_x(S+\pi_0)-\beta_0\pi_x^2}{T(T+L)^{\frac{3}{2}}\pi_0}\right]+\frac{hD}{2}$$

$$+\frac{hA\sigma}{2(T+L)^{\frac{1}{2}}}-\frac{hA\sigma}{4(T+L)^{\frac{3}{2}}}$$

$$+h\sigma\psi(A)\left[\frac{1}{T(T+L)^{\frac{1}{2}}}-\frac{1}{4(T+L)^{\frac{3}{2}}}-\frac{2\beta_0\pi_x}{\pi_0T(T+L)^{\frac{1}{2}}}+\frac{2\beta_0\pi_x}{(T+L)^{\frac{3}{2}}\pi_0}\right]$$

$$+\sigma\psi(A)\left[\frac{3\beta_0\pi_x^2}{4(T+L)^{\frac{5}{2}}\pi_0}-\frac{3\beta_0\pi_x(S+\pi_0)}{4(T+L)^{\frac{5}{2}}\pi_0}-\frac{3T\beta_0\pi_x}{4(T+L)^{\frac{5}{2}}\pi_0}\right]>0 \tag{B.8}$$

If $\beta_0\pi_x(S+\pi_0)-\beta_0\pi_x^2>0$, this implies $\pi_x<(S+\pi_0)$ $(\therefore \pi_x<\pi_0 \; and \; s>0)$ which holds.

$\therefore \frac{\partial^2 EAC}{\partial T^2}>0$ i.e. $H_{11}>0$

Similarly

$$\therefore=\left\{\begin{array}{l}\frac{2(k+\sum c_j)}{T^3}+\sigma\psi(A)\left[\begin{array}{l}\frac{3\beta_0\pi_x^2}{4\pi_0(T+L)^{\frac{5}{2}}}+\frac{2(S+\pi_0)}{T^3(T+L)^{\frac{-1}{2}}}\\-\frac{(S+\pi_0)}{T^2(T+L)^{\frac{1}{2}}}-\frac{(S+\pi_0)}{4T(T+L)^{\frac{3}{2}}}\\-\frac{3\beta_0\pi_x(S+\pi_0)}{4(T+L)^{\frac{5}{2}}\pi_0}\end{array}\right]\\-\frac{hA\sigma}{4(T+L)^{\frac{3}{2}}}\\+h\sigma\psi(A)\left[\frac{\beta_0\pi_x}{(T+L)^{\frac{3}{2}}\pi_0}-\frac{1}{4(T+L)^{\frac{3}{2}}}-\frac{3T\beta_0\pi_x}{4(T+L)^{\frac{5}{2}}\pi_0}\right]\end{array}\right\}\times\frac{2\sigma\psi(A)\beta_0}{\pi_0\sqrt{T+L}}-\left\{\begin{array}{l}\sigma\psi(A)\left(-\frac{\beta_0\pi_x}{(T+L)^{\frac{3}{2}}\pi_0}+\frac{\beta_0(S+\pi_0)}{2(T+L)^{\frac{3}{2}}\pi_0}\right)\\+h\sigma\psi(A)\beta_0\left(\frac{T}{2(T+L)^{\frac{3}{2}}\pi_0}-\frac{1}{(T+L)^{\frac{1}{2}}\pi_0}\right)\end{array}\right\}^2>0 \tag{B.9}$$

for $\pi_x<(S+\pi_0)$

Hence Proved.