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An optimization of an inventory model of decaying-lot depleted by declining market demand and extended with discretely variable holding costs

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C H R O N I C L E

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A B S T R A C T

Inventory management is considered as major concerns of every organization. In inventory holding, many steps are taken by managers that result a cost involved in this row. This cost may not be constant in nature during time horizon in which perishable stock is held. To investigate on such a case, this study proposes an optimization of inventory model where items deteriorate in stock conditions. To generalize the decaying conditions based on location of warehouse and conditions of storing, the rate of deterioration follows the Weibull distribution function. The demand of fresh item is declining with time exponentially (because no item can always sustain top place in the list of consumers' choice practically e.g. FMCG). Shortages are allowed and backlogged, partially. Conditions for global optimality and uniqueness of the solutions are derived, separately. The results of some numerical instances are analyzed under various conditions.

1. Introduction

One of the most important concerns of inventory management is to decide when and how much to order so that the total cost associated with the inventory system can be kept at minimum level. When inventory is decaying in nature, it becomes more important since deterioration cannot be ignored. There are various studies in this direction in continuous modification of inventory model for decaying items by including more and more practical features. Researchers are engaging in analyzing inventory models for deteriorating items such as volatile liquids, medicines, electronic components, fashion goods, fruits, vegetables, etc. An order level inventory model with constant deterioration was first developed by Aggarwal (1978).

Now, the inclusion of deterioration aspect into the inventory concept is incorporated in wide range of considered business environments in contemporary inventory models. Sana (2010) studied optimal

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selling price and lot size with time varying deterioration and partial backlogging. In this effort, an EOQ model over an infinite time horizon for perishable item where demand is price reliant and partial backorder permitted is discussed. Liao and Huang (2010) developed a deterministic inventory model for deteriorating items with trade credit financing and capacity constraints. They offered an inventory model for optimizing the replenishment cycle time for a single deteriorating item under a permissible delay in payments and constraints on warehouse capacity. Hung (2011) urbanized an inventory model with generalized type demand, deterioration and backorder rates. Bhunia and Shaikh (2011) developed a deterministic model for deteriorating items with displayed inventory level dependent demand rate incorporating marketing decisions with transportation cost. Khanra et al. (2011) offered an EOQ model for a deteriorating item with time–dependent quadratic demand under permissible delay in payment. In this study, a step was taken to analyze an EOQ model for deteriorating item considering quadratic time dependent demand rate and permissible delay in payment.

In various situations of inventory control, demand before ending spell exists and the inventory has mostly consumed through joint effect of the demand and the deterioration. This type of situations laid the foundation of supply out phenomena. Consequently, when supply out state occurs, some clients are willing to wait for backorder and others may wish to buy from supplementary sellers. Many researchers such as Park (1982), Hollier and Mak (1983) and Wee (1995) well thought-out the constant partial backlogging rates during the shortage period in their inventory models. In most inventory systems, the length of the waiting time for the next replenishment would come to a decision whether the backlogging will be accepted or not. Therefore, the backlogging rate is variable and dependent on the waiting time for the next replenishment. Chang and Dye (1999) investigated an EOQ model allowing shortage and partial backlogging. They assumed in their inventory model that the backlogging rate was variable and dependent on the length of the waiting time for the next replenishment. Many researchers modified inventory policies by considering the ‘‘time-proportional partial backlogging rate’’ such as Abad (2000), Papachristos and Skouri (2000), Wang (2002), Papachristos and Skouri (2003), etc.

Teng et al. (2003) then unmitigated the fraction of unsatisfied demand back ordered to any decreasing function of the waiting time up to the next replenishment. Teng and Yang (2004) widespread the partial backlogging EOQ model to allow for time-varying purchase cost. Yang (2005) prepared a comparison among various partial backlogging inventory lot size models for deteriorating stuffs on the basis of maximum profit. Teng et al. (2007) compared two pricing and lot sizing model for deteriorating objects with shortages. Dye et al. (2007) urbanized inventory and pricing strategies for deteriorating items with shortages. Skouri et al. (2011) projected an inventory model with general ramp type demand rate, constant deterioration rate, partial backlogging of unfulfilled demand and conditions of permissible delay in payments. Other related articles on inventory system with partial backlogging and shortages have been performed by Hou (2006), Jaggi et al. (2006, 2012), Patra et al. (2010), Yang et al. (2010), Lin (2012), Taleizadeh et al. (2011, 2012), etc.

However, a few number of researchers paid their attention towards generalizing the term of holding cost into the inventory models. Therefore, there are few literatures of inventory controlling phenomena under the aspect of variable holding cost. As alarmed above, most researchers unspecified that holding cost rate per unit time is invariable. However, more sophisticated storeroom facilities and services may be required for holding perishable items if they are kept for longer time. Therefore, in holding of perishable items, the assumption of unvarying holding cost rate is not always apt. Weiss (1982) noted that variable holding costs are suitable when the value of an item decreases the longer it is in stock. Ferguson et al. (2007) indicated that this type of model is suitable for perishable items in which price markdowns or removal of aging product are necessary. Alfares (2007) also assumed an inventory model with discretely variable holding cost. Recently, Mishra and Singh (2011) developed the inventory model for deteriorating items with time dependent linear demand and holding cost.

To give attention on the concept of variability of the holding cost of decaying item, Tyagi et al. (2012) developed an inventory model for decaying item with power demand pattern and managed first Weibull function for holding cost rate. In that study, the holding cost depends continuously on deterioration cost and storage period, shortages were allowed and partially backlogged inversely with the waiting time for the next replenishment. Therefore, this study has left a clear vacuum for study of the discrete change in the holding cost under considering environment of inventory set-ups. Tripathi (2013) studied an inventory model for time varying demand and constant demand; and time dependent holding cost and constant holding cost for case 1 and case2 respectively. He considered non-decaying items in his model and give a motivation to study our model for deteriorating items with discrete holding cost.

In result, an Economic Order Quantity (EOQ) inventory model of deteriorating item is considered with continuosly declining market demand. To extend such EOQ model in above mentioned directions, it is assumed that the holding cost rate per unit per unit time is discrete variable with respect to time and the deterioration rate of item is considered as two-parameter Weibull distributive function. Partial backlogging is allowed. The backlogging rate is an exponentially decreasing function of the waiting time for the next replenishment.

In this study, the primary problem is to minimize the average total cost per unit time by optimizing the shortage point per cycle. Separateing for each scenario, we show that minimized objective function is convex and the optimal solution is uniquely determined. Numerical example is proposed to illustrate the model and the solution procedure for each scenario of holding cost. The sensitivity analysis of major parameters is separately performed.

2 Notations

The following notations are used throughout the whole chapter

$I(t)$	Inventory level at any time $t, t \geq 0$;
T	Constant prescribed scheduling period or cycle length (time units);
$I_{\max}$	Maximum inventory level at the start of a cycle (units);
S	Maximum amount of demand backlogged per cycle (units);
t_1	Duration of inventory cycle when there is positive inventory;
Q	Order quantity (units/cycle);
c_1	Cost of the inventory items ($);
c_2	Fixed cost per order ($/order);
c_3	Shortage cost per unit back-ordered per unit time ($/unit/unit time);
c_4	Opportunity cost due to lost sales ($/unit).
$ATC_i(t_1^*)$	Average total cost per unit time in the i-th scenario, where $i = 1, 2$.

3. Assumptions

In developing the mathematical model of the inventory system, the following assumptions are made:

1. Replenishment rate is infinite;
2. Lead time is negligible;
3. The replenishment quantity and cycle length are constant for each cycle;
4. There is no replacement or repair of deteriorated items during a given cycle;
5. The time to deterioration of the item is Weibull dispersed. So, the rate of deterioration $d(t) = \alpha\beta t^{\beta-1}$, where α and β are shape and scale parameters;

6. The demand rate $R_1(t)$ is known and decreases exponentially as $R_1(t) = De^{-\lambda t}$ for $I(t) > 0$ and $R_1(t) = D$ for $I(t) \le 0$ where $D(>0)$ is initial demand and $0 < \lambda << 1$ is a constant governing the decreasing rate of the demand;
7. Shortages are permitted. Unfulfilled demand is partially backlogged. The backlogging rate $B(t)$ which is a decreasing function of the waiting time t for next replenishment, we here assume that $B(t) = e^{-\delta t}$, where $\delta \ge 0$, and t is the waiting time.

4. Model Formulations

As depicted above, the inventory arrangement goes like this: At $t = 0$, opening replenishment Q units are made, in which S units are delivered towards backorders, leaving a balance of I_{max} units in the initial inventory. From $t = 0$ to $t = t_1$ time units, the inventory level depletes owing to both demand and deterioration. At t_1, the inventory level is zero. During the time $(T - t_1)$ part of the shortage is backlogged and part of it is lost sales. Only the backlogging items are replaced by the after that replenishment.

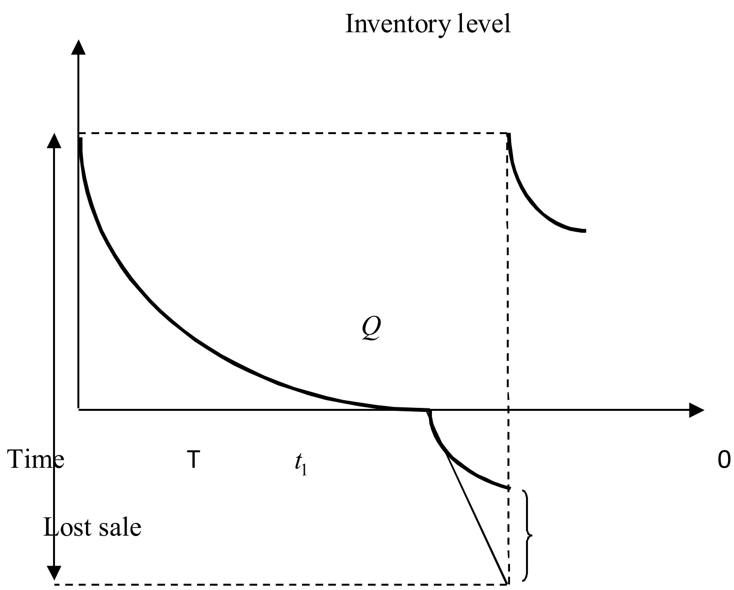

Fig. 1. Inventory system of decaying item for declining market demand

The inventory function with respect to time can be determined by evaluating the differential equations

$$\frac{dI(t)}{dt} + d(t)I(t) = -R_1(t) \quad 0 \le t \le t_1 \tag{1}$$

$$\frac{dI(t)}{dt} = -DB(t) \qquad t_1 \le t \le T \tag{2}$$

And with boundary conditions $I(0) = I_{max}$ and $I(t_1) = 0$. The approximate solution of Eq. (1) by neglecting higher order term of α is

$$I(t) = D\left[(t_1 - t) - \lambda\left(\frac{t_1^2}{2} - \frac{t^2}{2}\right) + \frac{\alpha}{\beta+1}\left(t_1^{\beta+1} - t^{\beta+1}\right)\right]e^{-\alpha t^{\beta}}; \qquad 0 \le t \le t_1 \tag{3}$$

Now, again taking the first two terms of the exponential series and neglecting the terms containing α^2 Eq. (4) becomes

$$I(t) = D\left[(t_1 - t) - \lambda\left(\frac{t_1^2}{2} - \frac{t^2}{2}\right) + \frac{\alpha}{\beta+1}\left(t_1^{\beta+1} - t^{\beta+1}\right)\right]\left(1 - \alpha t^{\beta}\right); \qquad 0 \le t \le t_1 \tag{4}$$

So, the maximum inventory level for each cycle can be obtained as

$$I_{\max} = I(0) = I(t) = D\left[t_1 - \frac{\lambda t_1^2}{2} + \frac{\alpha t_1^{\beta+1}}{\beta+1}\right] \tag{5}$$

During the shortage interval $[t_1, T]$, the demand at time t is partially backlogged at the fraction $B(t) = e^{-\delta t}$ Thus, the solution of differential Eq. (2) governing the amount of demand backlogged is as below

$$I(t) = -\frac{D}{\delta}\left[e^{-\delta(T-t)} - e^{-\delta(T-t_1)}\right], \qquad t_1 \le t \le T \tag{6}$$

with the boundary condition $I(t_1) = 0$. Let $t = T$ in Eq. (6), we obtain the maximum amount of demand backlogged per cycle as follows.

$$S = -I(T) = \frac{D}{\delta}\left[1 - e^{-\delta(T-t_1)}\right]. \tag{7}$$

Hence, the order quantity per cycle is given by

$$Q = I_{\max} + S = D\left(t_1 - \frac{\lambda t_1^2}{2} + \frac{\alpha t_1^{(1+\beta)}}{(1+\beta)}\right) + \frac{D}{\delta}\left[1 - e^{-\delta(T-t_1)}\right] \tag{8}$$

The order cost per cycle is

$$OC = c_2. \tag{9}$$

The deterioration cost per cycle is

$$DC = \int_0^{t_1} c_1 \alpha\beta t^{\beta-1} I(t)\,dt = c_1 D\alpha\left[\frac{t_1^{(1+\beta)}}{(1+\beta)} - \frac{\lambda t_1^{(2+\beta)}}{(2+\beta)}\right]. \tag{10}$$

The shortage cost per cycle is

$$SH = \int_{t_1}^{T} c_3(-I(t))\,dt = \frac{Dc_3}{\delta}\left[\frac{\left(1 - e^{-\delta(T-t_1)}\right)}{\delta} - (T - t_1)e^{-\delta(T-t_1)}\right] \tag{11}$$

The opportunity cost per cycle is

$$OPC = \int_{t_1}^{T}\left[1 - e^{-\delta(T-t)}\right]D\,dt = c_4 D\left[(T - t_1) - \frac{\left(1 - e^{-\delta(T-t_1)}\right)}{\delta}\right] \tag{12}$$

4.1 Holding Cost

Holding of inventory is a central part of inventory controlling phenomena. When item in collection has a deteriorating nature, it is more to be concerned of such items in stock holding. The owners of inventory have to endow not only for holding such item's units but also invest in handling these items for guardianship in good conditions. We are fascinated by this aspect to demonstrate a mathematical inventory model that can give us a picture which is better and very near to realities of business upbringing. Therefore, here we have understood that the holding cost of inventory is not constant and always depends upon time for which it has held. Now, here holding cost is measured as discretely variable holding cost with storage period. For using these assumptions, we have considered first two scenarios for discrete nature of variability of holding cost as retroactively variable holding cost and incrementally variable holding cost as:

Scenario 1: Retroactive holding cost;
Scenario 2: Incremental holding cost;

4.1.1 Scenario 1: Retroactive Holding Cost

In this scenario, the unit holding cost per unit time is well thought-out as discrete in nature, and increases as the time in storage increases, $h_1 < h_2 < h_3 < ... < h_n$, for storage periods 1 through n, respectively. A retroactive holding cost implies that the holding cost of the last storage period is applied retroactively to all previous periods in the order cycle. That is, if the cycle length is μ_1 or less, the unit holding cost is h_1 per time period; if the cycle length is between $\mu_1 < t \le \mu_2$, all inventory (retroactively) is charged a holding cost of h_2 per unit per time period; etc. Since the same holding cost will be applied to all units in the cycle, we only need to determine the total inventory level for the entire order cycle:

$$q = \int_0^{t_1} I(t)\,dt .$$

Therefore, holding cost is

$$HC = h_i \int_0^{t_1} I(t)\,dt = h_i D\left[\frac{t_1^2}{2} - \frac{\lambda t_1^3}{3} + \frac{\alpha\beta t_1^{\beta+1}}{(1+\beta)(\beta+2)} + \frac{\alpha\lambda t_1^{\beta+3}}{(1+\beta)(\beta+3)}\right] \tag{13}$$

where h is the corresponding value of $h = h_i$ for $\mu_{i-1} < t \le \mu_i$. Thus, the average total cost $ATC_1(t_1)$ of inventory cycle is

$$ATC_1(t_1) = [OC + HC_1 + DC + SC + OPC]/T$$

$$\begin{aligned} ATC_1(t_1) = \frac{D}{T}\Bigg[& h_i\left[\frac{t_1^2}{2} - \frac{\lambda t_1^3}{3} + \frac{\alpha\beta t_1^{\beta+1}}{(1+\beta)(\beta+2)} + \frac{\alpha\lambda t_1^{\beta+3}}{(1+\beta)(\beta+3)}\right] + \frac{c_2 T}{D} \\ & + c_1\alpha\left[\frac{t_1^{(1+\beta)}}{(1+\beta)} - \frac{\lambda t_1^{(2+\beta)}}{(2+\beta)}\right] + \frac{c_3}{\delta}\left[\frac{\left(1-e^{-\delta(T-t_1)}\right)}{\delta} - (T-t_1)e^{-\delta(T-t_1)}\right] \\ & + c_4 D\left[(T-t_1) - \frac{\left(1-e^{-\delta(T-t_1)}\right)}{\delta}\right]\Bigg] \end{aligned} \tag{14}$$

In the first scenario, the objective is to determine the optimal values of shortage point t_1 in order to minimize the average total cost $ATC_1(t_1)$ per unit time. The optimal solutions t_1^* need to satisfy the following equation.

$$\frac{dATC_1(t_1)}{dt_1} = \frac{D}{T} f_1(t_1) = 0, \tag{15}$$

where

$$f_1(t_1) = h_i\left[t_1 - \lambda t_1^2 + \frac{\alpha\beta t_1^{\beta+1}}{(1+\beta)} + \frac{\alpha\lambda t_1^{\beta+2}}{(1+\beta)}\right] + c_1\alpha\left[t_1^{\beta} - \lambda t_1^{\beta+1}\right] - \frac{\left(c_4\delta - c_3 e^{-\delta(T-t_1)}\right)}{\delta}, \tag{16}$$

$$-(T-t_1)c_3e^{-\delta(T-t_1)}-\frac{(c_3-\delta c_4)e^{-\delta(T-t_1)}}{\delta}.$$

Theorem 1 *If* $1>\delta T$, $1>>\lambda\geq 0$ *and* $\beta>1$ *then the solutions to Eq. (15) not only exists but also is unique (i.e., the optimal values* t_1^* *is uniquely determined).*

Proof: From (15), it is easily verified that, when $\delta T<1$ and $1>>\lambda\geq 0$ $\lim_{t_1\to 0} f_1(t_1)<0$ and $\lim_{t_1\to T} f_1(t_1)>0$. Furthermore, taking first derivative of $f_1(t_1)$ with respect to $t_1\in(0,T)$, we get $df_1(t_1)/dt_1>0$ .So, $f_1(t_1)$ is a strictly increasing function of $t_1\in(0,T)$. It implies that the (15) is verified at $t_1=t_1^*$, with $0<t_1^*<T$, which is the unique root of $f_1(t_1)=0$. This completes the proof.

Theorem 2 *If* $1>\delta T$,$1>>\lambda\geq 0$ *and* $\beta>1$ *the average total cost per unit time* $ATC_1(t_1)$ *is convex and reaches its global minimum at point* t_1^*.

Proof: From Eq. (15), if, $1>\delta T$, $1>>\lambda\geq 0$ we have

$$\left.\frac{d^2ATC_1(t_1)}{dt_1^2}\right|_{t_1=t_1^*}=\frac{D}{T}\left[f_1'(t_1)\right]\Big|_{t_1=t_1^*}>0.$$ It implies, t_1^* corresponds to the global minimum of convex $ATC_1(t_1)$. This completes the proof.

In this scenario, by using t_1^*, we can obtain the optimal maximum inventory level and the minimum average total cost per unit time from Eq. (5) and Eq. (14), respectively (we denote these values by $I_{\max}$ and $ATC_1(t_1^*)$). Furthermore, we can also obtain the optimal order quantity (we denote it by Q^*) from Eq. (8).

4.1.2 Scenario 2: Incremental Holding Cost

In this scenario, the discrete incremental unit holding cost increases as the time in storage increases. In this situation, though, an incremental holding cost implies that the holding cost of each storage period is applied only to the units apprehended during that period. That is, if the positive inventory time length is μ_1 or less, the unit holding cost is h_1 per time period; if the storage time-span is between $\mu_1<t_1\leq\mu_2$, the holding cost of h_1 is applied to the average inventory during the storage period from 0 to μ_1 and h_2 is applied from μ_1 to t_1; etc. Thus, we require evaluating the average inventory level for each storage phase within the order cycle (note, for the last storage period, μ_i is replaced with t_1):

$$q_i=\frac{D}{(\mu_i-\mu_{i-1})}\int_{\mu_{i-1}}^{\mu_i} D\left[(t_1-t)-\lambda\left(\frac{t_1^2}{2}-\frac{t^2}{2}\right)+\frac{\alpha}{\beta+1}\left(t_1^{\beta+1}-t^{\beta+1}\right)\right]\left(1-\alpha t^{\beta}\right)dt.$$

Therefore, holding cost per cycle is

$$\begin{aligned}HC_2&=\sum_{i=1}^{m}h_i(\mu_i-\mu_{i-1})q_i\\&=\sum_{i=1}^{m}h_iD\left[(\mu_i-\mu_{i-1})\left(t_1-\frac{\lambda t_1^2}{2}+\frac{\alpha t_1^{\beta+1}}{(\beta+1)}\right)-\frac{\alpha(\mu_i^{\beta+1}-\mu_{i-1}^{\beta+1})}{(\beta+1)}\left(t_1-\frac{\lambda t_1^2}{2}\right)\right.\\&\left.+\frac{\alpha\beta\left(\mu_i^{\beta+2}-\mu_{i-1}^{\beta+2}\right)}{(\beta+1)(\beta+2)}-\frac{\left(\mu_i^2-\mu_{i-1}^2\right)}{2}+\frac{\lambda\left(\mu_i^3-\mu_{i-1}^3\right)}{6}-\frac{\alpha\lambda\left(\mu_i^3-\mu_{i-1}^3\right)}{2(\beta+3)}\right].\end{aligned}\tag{17}$$

Thus, the average total cost $ATC_2(t_1)$ per unit time of inventory cycle is

$$ATC_2(t_1) = [OC + HC_2 + DC + SC + OPC]/T$$

$$\begin{aligned} ACT_2(t_1) = \frac{1}{T}\Bigg[\sum_{i=1}^{m} h_i \Bigg[(\mu_i - \mu_{i-1})\left(t_1 - \frac{\lambda t_1^2}{2} + \frac{\alpha t_1^{\beta+1}}{(\beta+1)}\right) \\ - \frac{\alpha(\mu_i^{\beta+1} - \mu_{i-1}^{\beta+1})}{(\beta+1)}\left(t_1 - \frac{\lambda t_1^2}{2}\right) + \frac{\alpha\beta\left(\mu_i^{\beta+2} - \mu_{i-1}^{\beta+2}\right)}{(\beta+1)(\beta+2)} - \frac{\left(\mu_i^2 - \mu_{i-1}^2\right)}{2} \\ + \frac{\lambda\left(\mu_i^3 - \mu_{i-1}^3\right)}{6} - \frac{\alpha\lambda\left(\mu_i^3 - \mu_{i-1}^3\right)}{2(\beta+3)}\Bigg] + \frac{c_2 T}{D} + c_1\alpha\left[\frac{t_1^{(1+\beta)}}{(1+\beta)} - \frac{\lambda t_1^{(2+\beta)}}{(2+\beta)}\right] \\ + \frac{c_3}{\delta}\left[\frac{\left(1 - e^{-\delta(T-t_1)}\right)}{\delta} - (T-t_1)e^{-\delta(T-t_1)}\right] + c_4 D\left[(T-t_1) - \frac{\left(1-e^{-\delta(T-t_1)}\right)}{\delta}\right]\Bigg]. \end{aligned} \tag{18}$$

In this scenario, the objective is to determine the optimal values of shortage point t_1 in order to minimize the average total cost $ATC_2(t_1)$ per unit time. The optimal solutions t_1^* need to satisfy the following equation.

$$\frac{dATC_2(t_1)}{dt_1} = \frac{D}{T} f_2(t_1) = 0, \tag{19}$$

where

$$\begin{aligned} f_2(t_1) = \sum_{i=1}^{e} h_i \left[\left(1 - \lambda t_1 + \alpha t_1^{\beta}\right)(\mu_i - \mu_{i-1}) - \frac{\alpha(1-\lambda t_1)\left(\mu_i^{\beta+1} - \mu_{i-1}^{\beta+1}\right)}{(\beta+1)}\right] \\ + c_1\alpha\left[t_1^{\beta} - \lambda t_1^{\beta+1}\right] - \frac{\left(c_4\delta - c_3 e^{-\delta(T-t_1)}\right)}{\delta} - (T-t_1)c_3 e^{-\delta(T-t_1)} - \frac{(c_3 - \delta c_4)e^{-\delta(T-t_1)}}{\delta} \end{aligned} \tag{20}$$

Theorem 3 *If* $\sum_{i=1}^{e} h_i(\mu_i - \mu_{i-1}) < \sum_{i=1}^{e} \frac{h_i\alpha\left(\mu_i^{\beta+1} - \mu_{i-1}^{\beta+1}\right)}{(\beta+1)} + c_4\left(1 - e^{-\delta T}\right) + c_3 T e^{-\delta T}$ *and* $1 >> \lambda > 0$, *then the solutions to Eq. (19) not only exists but also is unique (i.e., the optimal values* t_1^* *is uniquely determined).*

Proof: From Eq. (19), it is easily verified that, when $\sum_{i=1}^{e} h_i(\mu_i - \mu_{i-1}) < \sum_{i=1}^{e} \frac{h_i\alpha\left(\mu_i^{\beta+1} - \mu_{i-1}^{\beta+1}\right)}{(\beta+1)} + c_4\left(1 - e^{-\delta T}\right) + c_3 T e^{-\delta T}$ and $1 >> \lambda > 0$ $\lim_{t_1 \to 0} f_2(t_1) < 0$ and $\lim_{t_1 \to T} f_2(t_1) > 0$. Furthermore, taking first derivative of $f_2(t_1)$ with respect to $t_1 \in (0,T)$, we get $df_2(t_1)/dt_1 > 0$. So, $f_2(t_1)$ is a strictly increasing function of $t_1 \in (0,T)$. It implies that the (19) is verified at $t_1 = t_1^*$, with $0 < t_1^* < T$, which is the unique root of $f_2(t_1) = 0$. This completes the proof.

Theorem 4 *If* $\sum_{i=1}^{e} h_i(\mu_i - \mu_{i-1}) < \sum_{i=1}^{e} \frac{h_i\alpha\left(\mu_i^{\beta+1} - \mu_{i-1}^{\beta+1}\right)}{(\beta+1)} + c_4\left(1 - e^{-\delta T}\right) + c_3 T e^{-\delta T}$ *and* $1 >> \lambda > 0$, *the average total cost per unit time* $ATC_2(t_1)$ *is convex and reaches its global minimum at point* t_1^*.

Proof: From Eq. (19), if $\sum_{i=1}^{e} h_i\,(\mu_i - \mu_{i-1}) < \sum_{i=1}^{e} \frac{h_i \alpha\left(\mu_i^{\beta+1} - \mu_{i-1}^{\beta+1}\right)}{(\beta+1)} + c_4\left(1-e^{-\delta T}\right) + c_3 T e^{-\delta T}$ and $1 >> \lambda > 0$ we have $\left.\frac{d^2 ATC_2(t_1)}{dt_1^2}\right|_{t_1=t_1^*} = \frac{D}{T}\left[f_2'(t_1)\right]\Big|_{t_1=t_1^*} > 0$. It implies, t_1^* corresponds to the global minimum of convex $ATC_2(t_1)$. This completes the proof. In this scenario, by using t_1^*, we can obtain the optimal maximum inventory level and the minimum average total cost per unit time $ATC_2(t_1^*)$ from (5) and (19), respectively. Furthermore, we can also obtain the optimal order quantity from (8).

5. Numerical Examples

As an illustration of both scenarios of developed model, a numerical example is presented for a single product. To perform the numerical analysis, data have been taken randomly from literatures in appropriate units.

Example 1: We consider an inventory system which verifies the described assumptions above. The input data of parameters are taken randomly as $T=4, a=0.4, b=2, \alpha=0.8, h_1=0.4, h_2=0.5, h_3=0.6$ $\beta=2, \mu_0=0, \mu_1=1, \mu_2=2, \mu_3=t_1$ $d=10, c_1=3, c_2=1, c_3=3, R=2, H=0.4$ and $c_4=2$.

By using MATHEMATICA 8.0, the global minimum Average Total Cost per unit time $ATC_i(t_1)$, $i=1,2$ along with the optimal value of t_1^* is calculated for each the proposed i-th scenario. The Optimal Order Quantity (Q^*) is also calculated in each scenario. The summary of crucial values for each scenario is given below.

Table 1
Summary of model's optimal values in i-th scenario

No. of scenario	t_1^*	Q^*	$ATC_i(t_1^*)$
1	1.543017	115.4670	344.737
2	1.584176	116.259	342.062

Observations: One can make following remarks.
i.The Optimal Average Total Cost per unit time is greater in the scenario 1.
ii.The Optimal Order Quantity has maximum value in the scenario 2.

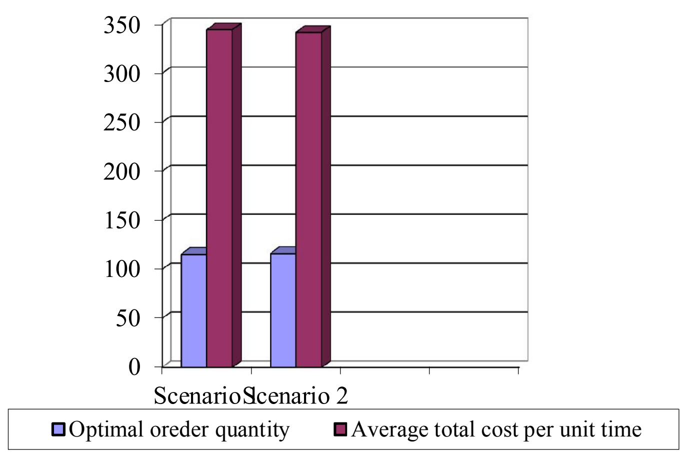

Fig. 2. Inventory model optimal values for each scenario

6. Sensitivity Analysis

In this section, the effects of studying the changes in the optimal value of Average Total Cost per unit time, the optimal shortage point and the optimal value of Order Quantity per cycle of each scenario with respect to changes in some model parameters are discussed. The sensitivity analysis in each scenario is performed by changing the value of each of the parameters by $\pm 5\%$ and $\pm 10\%$, taking one parameter at a time and keeping the remaining parameters unchanged. Example 1 is used in each scenario.

6.1 Sensitivity Analysis for Scenario 1

To discuss the effect of changes of model parameters $T, h_1, \alpha, \beta, \lambda, c_1, c_3, c_4$ and δ on the optimal value of the average total cost $(ATC_1(t_1^*) = 344.737)$, the shortage time point $(t_1^* = 1.543017)$ and the value of Order Quantity per cycle $(Q^* = 115.4670)$ for scenario 1, the different values of these parameter according to $\pm 5\%$ and $\pm 10\%$ change in each have taken and its effect on $TAC_1(t_1^*)$, t_1^* and Q^* are presented in the following Table 2.

Table 2
Sensitivity Analysis for Scenario 1

Parameters	t_1^*	Q^*	$ATC_1(t_1^*)$	% change in the values of		
				t_1^*	Q^*	$ACT_1(t_1^*)$
$T=4$	1.619650	120.013	337.220	+4.97	+3.93	-2.18
	1.582259	117.837	341.002	+2.54	+2.05	-1.08
	1.501826	112.879	348.373	-2.67	-2.26	+1.05
	1.458579	110.048	351.851	-5.47	-4.69	+2.06
$h_1=0.4$	1.528986	115.283	345.823	-0.91	-0.16	+0.31
	1.535944	115.334	345.285	-0.45	-0.11	+0.16
	1.550208	115.604	344.179	+0.46	+0.11	-0.16
	1.557521	115.743	343.610	+0.94	+0.24	-0.32
$\alpha=0.8$	1.487130	115.309	349.062	-3.62	-0.14	+1.25
	1.514258	115.392	346.973	-1.86	-0.06	+0.65
	1.573582	115.533	342.334	+1.98	+0.06	-0.69
	1.606155	115.588	339.745	+4.09	+0.14	-1.44
$\beta=2$	1.493807	114.694	348.099	-3.18	-0.67	+0.97
	1.577331	115.069	346.501	-1.66	-0.34	+0.51
	1.571146	115.890	342.781	+1.82	+0.37	-0.57
	1.602041	116.338	340.605	+3.82	+0.75	-1.19
$\lambda=0.1$	1.552731	115.531	343.997	+0.63	+0.05	-0.21
	1.547837	115.499	344.370	+0.31	+0.03	-0.10
	1.538269	115.437	345.097	-0.31	-0.02	+0.10
	1.533590	115.407	345.452	-0.62	-0.05	+0.21
$c_1=3$	1.494772	114.571	348.519	-3.13	-0.77	+1.09
	1.518356	115.005	346.679	-1.59	-0.40	+0.56
	1.568832	115.962	342.682	+1.67	+0.42	-0.59
	1.595885	116.489	340.505	+3.43	+0.88	-1.22
$c_4=2$	1.547561	115.553	311.002	+0.29	+0.07	-9.78
	1.545253	115.510	327.870	+0.15	+0.03	-4.89
	1.540735	115.424	361.602	-0.15	-0.03	+4.89
	1.538447	115.381	378.467	-0.30	-0.07	+9.78
$c_3=3$	1.601166	116.594	407.422	+3.77	+0.98	+18.18
	1.572582	116.034	376.160	+1.92	+0.49	+9.11
	1.512403	114.895	313.154	-1.98	-0.49	-9.16
	1.480661	114.316	281.416	-4.04	-0.99	-18.37
$\delta=0.1$	1.532175	106.942	269.351	-0.70	-7.38	-21.87
	1.537597	111.009	303.841	-0.35	-3.86	-11.86
	1.547597	120.179	394.314	+0.30	+4.08	+14.38
	1.553851	125.179	452.195	+0.70	+8.41	+31.17

Observations: From Table 2 the following observations can be made as:

1. $ATC_1(t_1^*)$ increases with increase in the values of model parameters h_1, α, β, c_1 and c_3 while $ATC_1(t_1^*)$ decreases with increase in the value of T, λ, c_4 and δ. $ATC_1(t_1^*)$ is highly sensitive to changes in T, c_3, c_4 and δ. It is less sensitive to changes in α, β and c_1; and very less sensitive to change in h_1 and λ;

2. $ATC_1(t_1^*)$ decreases with decrease in the values of model parameters h_1, α, β, c_1 and c_3 while $ATC_1(t_1^*)$ increases with decrease in the value of T, λ, c_4 and δ. $ATC_1(t_1^*)$ is highly sensitive to changes in T, c_3, c_4 and δ. It is less sensitive to changes in α, β and c_1; and very less sensitive to change in h_1 and λ;

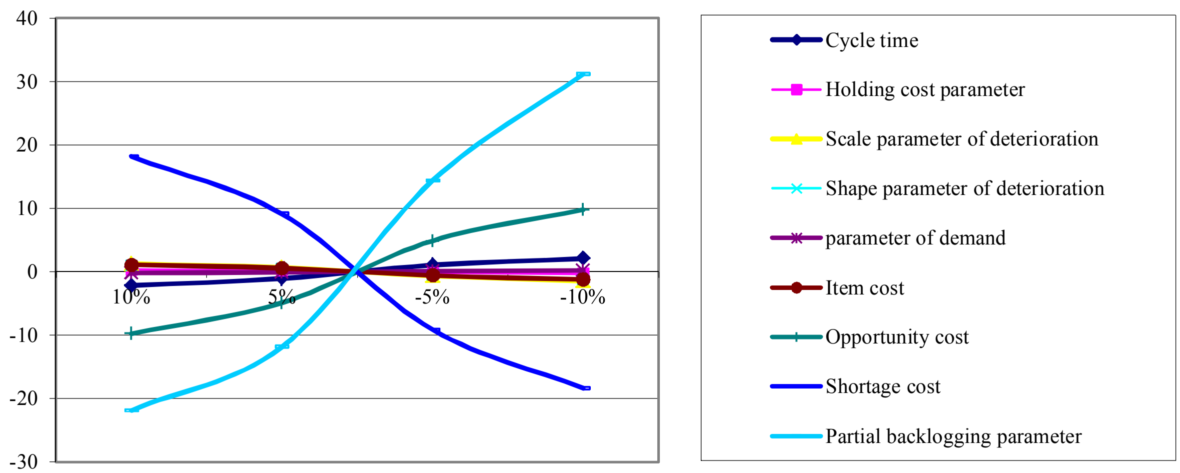

Fig. 3. Behavior of optimal average total cost per unit time in scenario 1

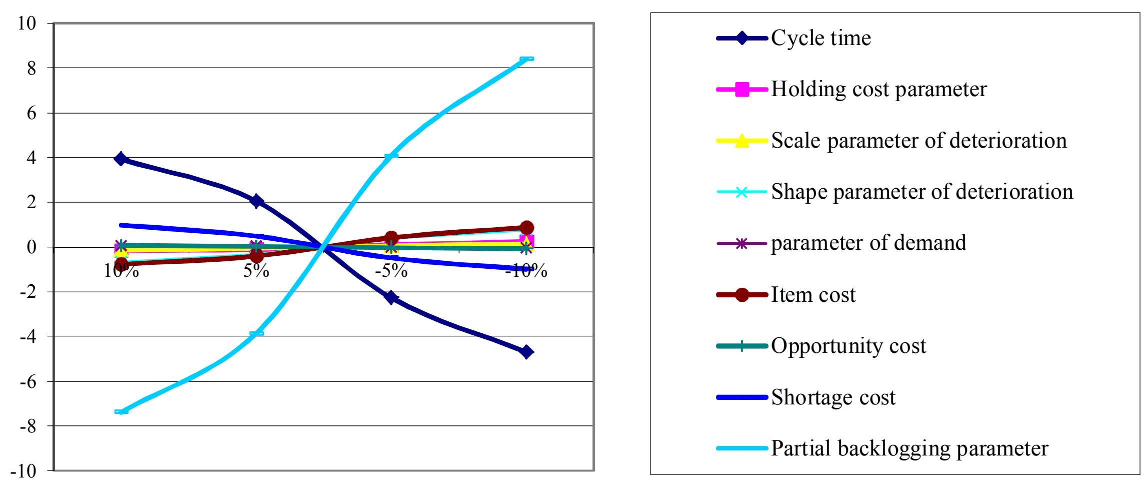

Fig. 4. Behavior of optimal ordering quantity in scenario 1

3. Q^* increases with increase in the values of model parameters T, λ, c_3 and c_4 while Q^* decreases with increase in the value of h_1, α, β, c_1 and δ. Q^* is highly sensitive to changes in T and δ. It is less sensitive to changes in h_1, β, c_1 and c_3; and very less sensitive to change in α, λ and c_4;

4. Q^* decreases with decrease in the values of model parameters T, λ, c_3 and c_4 while Q^* increases with decrease in the value of h_1, α, β, c_1 and δ. Q^* is highly sensitive to changes in T and δ. It is less sensitive to changes in h_1, β, c_1 and c_3; and very less sensitive to change in α, λ and c_4.

6.2 Sensitivity Analysis for Scenario 2

To discuss the effect of changes of model parameters $T, h_1, \alpha, \beta, \lambda, c_1, c_3, c_4$ and δ on the optimal value of the average total cost $(ATC_2(t_1^*) = 342.062)$, the shortage time point $(t_1^* = 1.584176)$ and the value of Order Quantity per cycle $(Q^* = 116.259)$ for scenario 2, the different values of these parameter according to $\pm 5\%$ and $\pm 10\%$ change in each have taken and its effect on $TAC_2(t_1^*)$, t_1^* and Q^* are presented in the following Table 3.

Table 3
Sensitivity Analysis for Scenario 2

Parameters	t_1^*	Q^*	$ATC_2(t_1^*)$	% change in the values of t_1^*	% change in the values of Q^*	% change in the values of $ACT_2(t_1^*)$
$T = 4$	1.669500	121.191	334.654	+5.38	+4.24	-2.16
	1.627751	118.817	338.377	+2.75	+2.20	-1.07
	1.538688	113.496	345.666	-2.87	-2.37	+1.05
	1.491191	110.504	349.133	-5.87	-4.95	+2.07
$h_1 = 0.4$	1.572252	116.027	343.028	-0.75	-0.19	+0.28
	1.578188	116.143	342.548	-0.37	-0.09	+0.14
	1.590217	116.378	341.571	+0.38	+0.10	-0.14
	1.596311	116.498	341.075	+0.76	+0.20	-0.29
$\alpha = 0.8$	1.521925	116.011	346.880	-3.93	-0.21	+1.41
	1.552087	116.138	344.659	-2.08	-0.10	+0.72
	1.618405	116.373	339.366	+2.16	+0.09	-0.79
	1.655021	116.476	336.443	+4.47	+0.18	-1.64
$\beta = 2$	1.527272	115.364	345.990	-3.59	-0.76	+1.14
	1.554366	115.797	344.132	-1.88	-0.39	+0.60
	1.617085	116.753	339.744	+2.07	+0.42	-0.68
	1.653559	117.279	337.135	+4.38	+0.88	-1.44
$\lambda = 0.1$	1.595859	116.362	341.178	+0.74	+0.08	-0.26
	1.589962	116.310	341.625	+0.36	+0.04	-0.13
	1.578497	116.211	342.491	-0.36	-0.04	+0.12
	1.572921	116.164	342.911	-0.71	-0.08	+0.24
$c_1 = 3$	1.529091	115.205	346.347	-3.47	-0.90	+1.25
	1.555909	115.713	344.272	-1.78	-0.46	+0.65
	1.614026	116.850	339.704	+1.88	+0.50	-0.69
	1.645610	117.491	337.179	+3.88	+1.05	-1.43
$c_4 = 2$	1.589090	116.356	308.276	+0.31	+0.08	-9.88
	1.586636	116.308	325.170	+0.15	+0.04	-4.93
	1.581708	116.211	358.954	-0.15	-0.04	+4.93
	1.579230	116.763	375.844	-0.31	-0.08	+9.87
$c_3 = 3$	1.647231	117.524	404.041	+3.98	+1.08	+18.12
	1.616221	116.894	373.145	+2.02	+0.55	+9.08
	1.551025	115.619	310.795	-2.09	-0.55	-9.14
	1.516686	114.974	279.847	-4.26	-1.10	-18.33
$\delta = 0.1$	1.572687	107.747	267.054	-0.72	-7.32	-21.92
	1.578434	111.808	301.370	-0.36	-3.82	-11.89
	1.589913	121.162	390.622	+0.36	+4.22	+14.19
	1.595645	126.592	448.987	+0.72	+8.88	+31.25

Observations: From Table 3 the following observations can be made as;

1. $ATC_2(t_1^*)$ increases with increase in the values of model parameters h_1, α, β, c_1 and c_3 while $ATC_2(t_1^*)$ decreases with increase in the value of T, λ, c_4 and δ. $ATC_2(t_1^*)$ is highly sensitive to changes in T, c_3, c_4 and δ. It is less sensitive to changes in α, β and c_1; and very less sensitive to change in h_1 and λ.

2. $ATC_2(t_1^*)$ decreases with decrease in the values of model parameters h_1, α, β, c_1 and c_3 while $ATC_2(t_1^*)$ increases with decrease in the value of T, λ, c_4 and δ. $ATC_2(t_1^*)$ is highly sensitive to changes in T, c_3, c_4 and δ. It is less sensitive to changes in α, β and c_1; and very less sensitive to change in h_1 and λ.

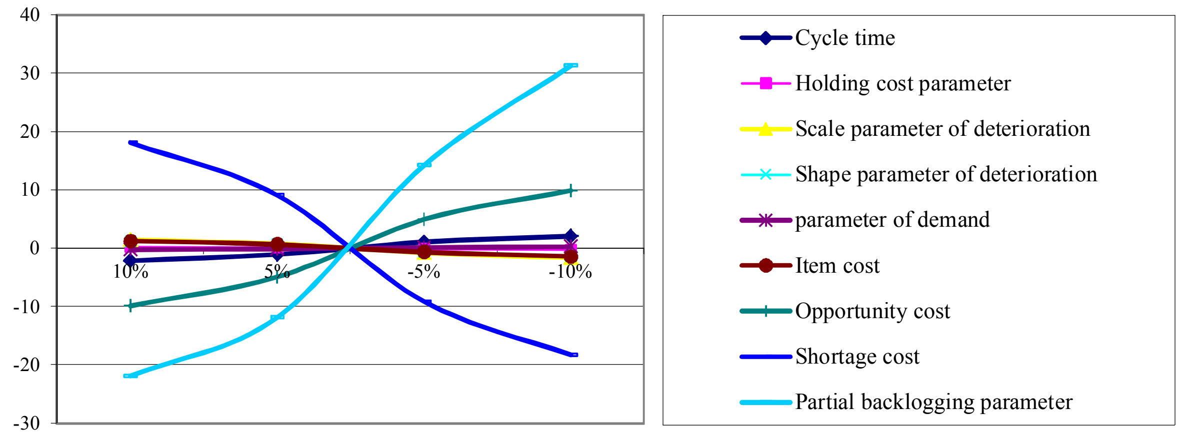

Fig. 5. Behavior of optimal average total cost per unit time in scenario 2

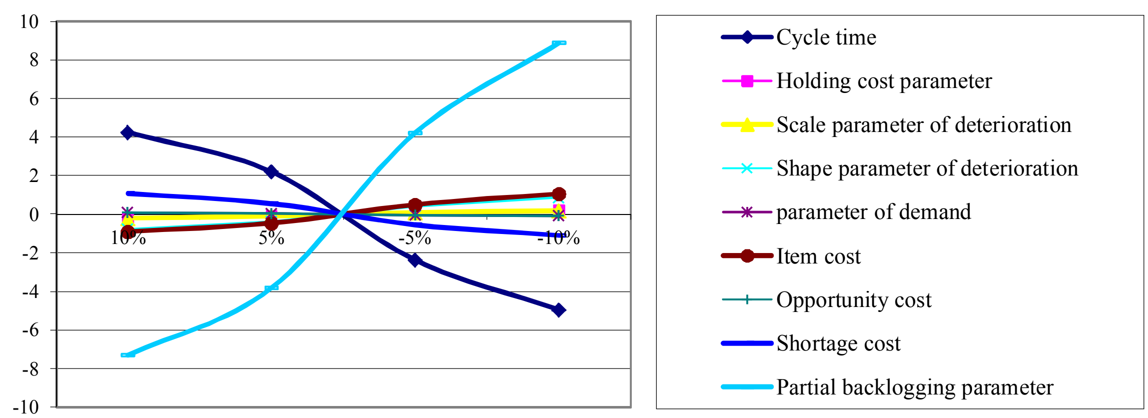

Fig. 6. Behavior of optimal ordering quantity in scenario 2

3. Q^* increases with increase in the values of model parameters T, λ, c_3 and c_4 while Q^* decreases with increase in the value of h_1, α, β, c_1 and δ. Q^* is highly sensitive to changes in T and δ. It is less sensitive to changes in h_1, β, c_1 and c_3; and very less sensitive to change in α, λ and c_4.

4. Q^* decreases with decrease in the values of model parameters T, λ, c_3 and c_4 while Q^* increases with decrease in the value of h_1, α, β, c_1 and δ. Q^* is highly sensitive to changes in T and δ. It is less sensitive to changes in h_1, β, c_1 and c_3; and very less sensitive to change in α, λ and c_4.

7. Conclusions

In this model, we have studied an inventory model in which the inventory is depleted not only by declining pattern of demand but also by Weibull distributed deterioration where holding cost per unit time is considered a discretely variable. Shortages are allowed and partially backlogged. Conditions for existence and uniqueness of the optimal solution have been provided. Therefore, the proposed model can be used widely in inventory-control of certain deteriorating items such as food items, electronic components, and fashionable commodities, and others. Moreover, the advantage of the proposed inventory model is illustrated with example. This study highlights that the optimal average total cost per unit time is high when holding cost per unit per unit time is considered as retroactively to all previous periods of storing and optimal value of ordered quantity is less. On the other hand, the optimal average total cost per unit time is less when holding cost per unit per unit time is considered as incremental to periods of storing and optimal value of ordered quantity is high. In future, this paper may be extended with stochastic demand and permissible delay of payment.

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