

A multi-item inventory system with expected shortage level-dependent backorder rate with working capital and space restrictions

A. Gholami-Qadikolaie, A. Mirzazadeh* and M. Kajizad

Department of Industrial Engineering, Tarbiat Moallem University, Tehran, Iran

ARTICLE INFO

Article history:

Received 1 June 2011
Received in revised form
July, 29, 2011
Accepted 30 July 2011
Available online
3 August 2011

Keywords:

Inventory system
Stochastic demand
Stochastic lead time
Partial backlogging
Chance-constrained method
Minimax distribution free
procedure

ABSTRACT

In this paper, a new multi-item inventory system is considered with random demand and random lead time including working capital and space constraints with three decision variables: order quantity, safety factor and backorder rate. The demand rate during lead time is stochastic with unknown distribution function and known mean and variance. Random constraints are transformed to crisp constraints with using the chance-constrained method. The Minimax distribution free procedure has been used to lead proposed model to the optimal solution. The shortage is allowed and the backlogging rate is dependent on the expected shortage quantity at the end of cycle. Two numerical examples are presented to illustrate the proposed solution method.

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1. Introduction

The shortage cost calculation is an important problem in estimating the inventory systems costs including purchasing, set up, holding, stock out costs and, etc. Shortage cost is divided to the backorder and lost sale. Furthermore, Lead time management is a significant issue in production and operation management. As stated in Tersine (1994), lead time usually comprises several components, such as set up time, waiting time, move time and queue time. In many practical situations, lead time can be reduced using an added crashing cost. In other words, lead time is controllable. Liao and Shyu (1991), Ben Daya and Rauf (1994), Ouyang et al. (1996), Park (2007) considered lead time as a variable and controlled it by paying extra crashing cost. They assumed that the lead time could be decomposed into n mutually independent components where each component has fixed crashing cost. Besides Callego and Moon (1993) assumed unfavorable lead time demand distribution and solved both the continuous and periodic review models with a mixture of backorder and lost sale using Minimax distribution free method.

There are some multi-product inventory models with shortage including restrictions on inventory investment, space or reorder work load. Brown and Gerson (1967) proposed some models for multi-

* Corresponding author Tel.: +98 21 4445 1990
E-mail: a.mirzazadeh@aut.ac.ir (A. Mirzazadeh)

item stochastic inventory system with the limitation on total inventory investment. Schrady and Choe (1971) proposed a model with the total time weighted shortages with limitations on inventory investment and reorder work load. Gardner (1983) developed models for minimizing expected approximate backordered sales with the restrictions on aggregate investment and replenishment work load. Schroeder (1974) presented a model constrained by total expected annual ordering with an objective of minimizing the expected number of unit's backordered per year.

In many real-world situations, during a shortage period, the longer the waiting time is, the smaller the backlogging rate is. For instance, for fashionable commodities and high-tech products with a product life cycle, the willingness for a customer to wait for backlogging is diminishing with the length of the waiting time. In this way, the researcher used β as the function of τ , the time remaining until the next replenishment. Montgomery et al. (1973) proposed linear function for $\beta(\tau)$. Abad (1996) introduced exponential $\beta(\tau)$ originally, but Papachristos and Skouri (2000) referred it as exponential. Abad (1996) proposed rational $\beta(\tau)$ for the first time and then San Jose et al. (2005) and Silica et al. (2007) used this form and the first to use this form to it. Silica et al. (2009) proposed mixed exponential $\beta(\tau)$ in their study. Some other authors considered β as a function of expected shortage quantity at the end of cycle. Their studies are based upon this assumption, which the larger amount of the expected shortage at the end of cycle, the smaller amount of customer can wait and hence the smaller backorder rate would be. Ouyang and Chung (2001) were first to introduce this assumption in their model and some other authors generalized this assumption in their models for backorder rate (Lee, 2005; Lee et al., 2007; Lee et al., 2006).

Many models for continuous inventory system with stochastic demand and allowable stock out such as Hadley and Whitin (1963), Parker (1964), Tinareli (1983) and Yano (1976) and some models in the stochastic demand and stochastic lead time environment such as Ord and Bagchi (2006), Burgin (2007) have been studied to find optimal solution on order quantity and backorder rate, which depends on expected shortage quantity and reorder point, which is replaced by safety factor.

Ouyang and Chung (2001) observed that many products of well-known brand and modish goods like certain brand gum shoes and clothes may lead to a state in which clients prefer their demands to be backordered, whereas shortage happened. Doubtlessly, if the amount of shortage exceeds the waiting patience of client, some clients avoid the backorder case. This phenomenon reveals that as shortage occurs, in the stochastic demand and deterministic lead time area, the longer the length of lead time is the larger amount of shortage is, the smaller proportion of customers can wait and hence the smaller backorder rate would be. However, in our new suggested model, in the stochastic lead time and stochastic demand environment, we consider safety factor, order quantity and backorder rate as the decision variables and assume that the backorder rate is dependent on the expected shortage quantity at the end of cycle. Thus, the larger amount of the safety factor is, the larger amount of safety stock is, the larger holding cost is, the smaller amount of expected shortage quantity through the stochastic lead time is, the larger back order rate is and therefore, the smaller stock out cost would be. Therefore, our suggested model balances holding cost and stock out cost to minimize the total expected cost (TECU).

In this paper, first we consider multi product inventory system with three variables (order quantity, backorder rate and safety factor) with space and working capital constraints. We assume that the probability distribution of demand during lead time is unknown, but mean and variance of demand are known. Then, we utilize Minimax distribution free method to minimize the total expected cost per unit time. We transform random working capital and random space constraints to crisp constraints with using the chance-constrained method and then, solve the problem with Lagrange's multiplier method. At the end of paper, we present two numerical examples to illustrate our solution procedure.

2. Notation and assumption

The following notations have been used in this paper:

Indices:

- n Number of items
 i Index of items

Parameters:

- π_{i1} Penalty cost per unit for i-th item
 π_{i2} Marginal profit per unit for i-th item
 c_i Purchasing cost per unit for i-th item
 f_i Space used per unit for i-th item
 A_i Ordering cost per order for i-th item
 θ_i Backorder parameter ($\theta_i \geq 0$) for i-th item
 B Maximum inventory investment for all items
 F Maximum available space for all items

Decision variables:

- Q_i Order quantity for i-th item
 R_i Reorder point (which is replace by safety factor) for i-th item
 k_i Safety factor for i-th item
 β_i The fraction of demand which is backordered during stockout period for i-th item

Random variables:

- D_i Demand rate per unit time for i-th item
 L_i Length of lead time for i-th item
 x_i Demand during lead time for i-th
 x^+ Maximum value of x and 0
 $E(\cdot)$ Mathematical expectation

The developed model is based on these assumptions:

- Shortage is allowed and partially backlogged.
- Demand rate D_i is a random variable with mean $E(D_i)$ and standard deviation σ_{D_i} .
- Lead time is randomly distributed with mean $E(L_i)$ and standard deviation σ_{L_i} .
- Demand during lead time x_i is convolution of the demand rate and lead time. If demand rate, D_i , and lead time, L_i , be independent to each other the mean and variance of x_i is (Tersine (1994)):

$$E(x_i) = E(D_i) \times E(L_i), \quad (1)$$

$$Var(x_i) = \sigma_{x_i}^2 = Var(D_i) \times E(L_i) + (E(D_i))^2 \times Var(L_i). \quad (2)$$

- The reorder point R_i is the expected demand during lead time plus safety stock (ss) and $ss = k_i \times$ (standard deviation of lead time) i.e. $R_i = E(x_i) + k_i \sigma_{x_i}$ where k_i is safety factor satisfying $P(x_i > R_i) = P(z_i > k_i) = \alpha_i$, z_i represents the standard normal random variable and α_i represents the allowable stockout probability during lead time $E(x_i)$.

- Inventory is continuously reviewed. The replenishments are made whenever the inventory level falls to the reorder point R_i .
- The purchasing cost for i -th item is paid at the time of order received.

3. Model formulation

The system manager places an order of amount Q_i for i -th item, when the inventory level reaches to reorder level. The expected demand during shortage at the end of cycle is:

$$E(x_i - R_i)^+ = \int_{R_i}^{\infty} (x_i - R_i) f(x_i) dx_i \tag{3}$$

The expected net inventory level at the end of cycle is calculated by (see Fig. 1):

$$\int_{-\infty}^{\infty} (R_i - x_i) f(x_i) \times dx_i = R_i \int_{-\infty}^{\infty} f(x_i) dx_i - \int_{-\infty}^{\infty} x_i f(x_i) = R_i - E(x_i) \tag{4}$$

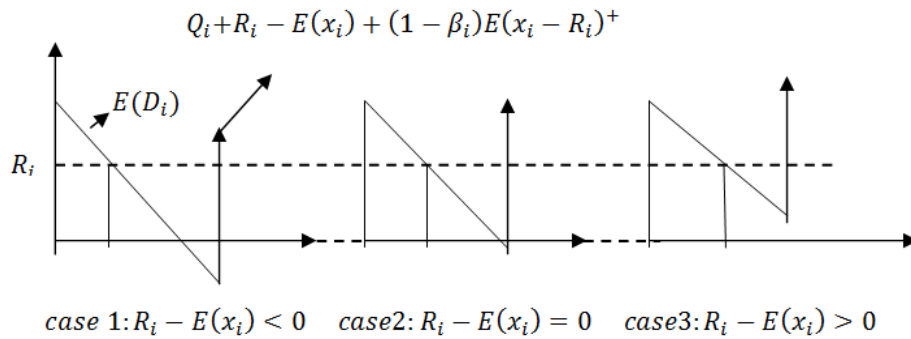


Fig. 1. The inventory situation when both demand and lead-time are random

So, the expected number of backorder at the end of cycle is $\beta_i E(x_i - R_i)^+$ and the expected number of lost sale at the end of cycle is $(1 - \beta_i) E(x_i - R_i)^+$. The expected net inventory level just before the order arrives is $R_i - E(x_i) + (1 - \beta_i) E(x_i - R_i)^+$ and the expected net inventory level at the beginning of the cycle is $Q_i + R_i - E(x_i) + (1 - \beta_i) E(x_i - R_i)^+$ and holding cost per cycle is calculated as follow (first we do not consider backorder rate):

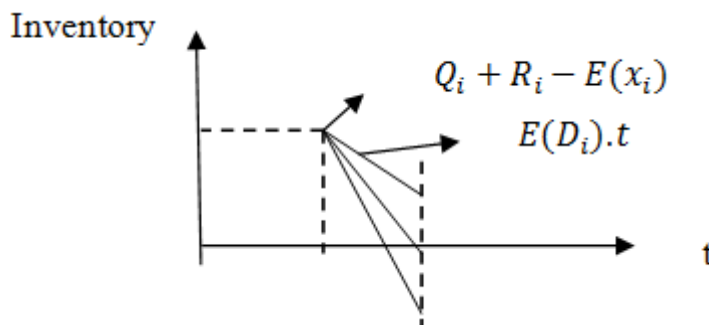


Fig. 2. Demand function during time

$$I(t) = (Q_i + R_i - E(x_i)) - E(D_i)t \quad , \quad T_i = \frac{Q_i}{E(D_i)} \tag{5}$$

$$\text{holding cost per cycle} = h_i \times \int_0^{T_i} [(Q_i + R_i - E(x_i)) - E(D_i)t] dt$$

$$\begin{aligned}
 &= \left[(Q_i + R_i - E(x_i)) \times T_i - E(D_i) \times \frac{T_i^2}{2} \right] \\
 &= \left[\frac{Q_i^2}{2E(D_i)} + \frac{Q_i}{E(D_i)} (R_i - E(x_i)) \right] \times h_i = \frac{Q_i}{E(D_i)} \left[\frac{Q_i}{2} + R_i - E(x_i) \right] \times h_i
 \end{aligned} \tag{6}$$

In the above equations, for calculating holding cost per cycle, we don't consider the fraction demand during stockout period which will be backordered (β). It means that all of the demands are backordered. If we consider (β), we have lost sales and holding cost per cycle will be changed as follow (see Appendix A):

$$\frac{Q_i}{E(D_i)} \left[\frac{Q_i}{2} + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+ \right] \times h_i \tag{7}$$

Thus, the mathematical model of the expected cost per cycle can be expressed by:

$$\begin{aligned}
 &\sum_i A_i + \frac{Q_i}{E(D_i)} \left[\frac{Q_i}{2} + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+ \right] \times h_i + \pi_{i1}\beta_i E(x_i - R_i)^+ + \\
 &\pi_{i2}(1 - \beta_i)E(x_i - R_i)^+
 \end{aligned} \tag{8}$$

Therefore, the total expected cost per unit time (TECU) is simply calculated by multiplying Eq. (8) in the expected number of cycle and model is transformed as follow:

$TECU(Q_i, R_i, \beta_i)$ = ordering cost + holding cost + stockout cost

$$\begin{aligned}
 TECU(Q_i, R_i, \beta_i) &= \sum_i \frac{E(D_i)}{Q_i} A_i + \left[\frac{Q_i}{2} + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+ \right] \times h_i \\
 &+ \frac{E(D_i)}{Q_i} [\pi_{i1}\beta_i E(x_i - R_i)^+ + \pi_{i2}(1 - \beta_i)E(x_i - R_i)^+]
 \end{aligned} \tag{9}$$

We consider backorder rate as a variable, which is dependent on the expected shortage quantity at the end of cycle. It means that when shortage occurs, the larger amount of shortage is, the smaller ratio of client can wait and therefore, the smaller backorder rate would be. Thus, backorder will be function of the expected shortage quantity, which can be expressed as follow:

$$\beta_i = \frac{1}{1 + \theta_i E(x_i - R_i)^+} \tag{10}$$

Backorder parameter (θ_i) is a positive constant, which exhibits the importance of shortage in calculating backorder rate (β_i). Our objective is to minimize total expected cost per unit time with two restrictions, working capital and space and our model is formulated as follow:

$$\min TECU(Q_i, R_i, \beta_i)$$

subject to

$$\begin{aligned}
 &\sum_{i=1}^n c_i [Q_i + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+] \leq B, \\
 &\sum_{i=1}^n f_i [Q_i + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+] \leq F.
 \end{aligned} \tag{11}$$

According to Tersine (1994), the distribution of demand is normal at the factory level; the Poisson at the retail level; and the exponential at wholesale and retail level. In addition, the distribution of lead time may be gamma, exponential geometric and normal. Bagchi et al. (1986) discussed elaborately on these topics. We relaxed the assumption on the distribution of demand rate lead time and demand during lead time with the following assumptions:

1. The distribution function $F(d_i)$ of D_i belong to the class of distribution function τ_{di} with finite mean μ_{D_i} and standard deviation σ_{D_i} .
2. The distribution function $G(l_i)$ of L_i belong to the class of distribution function τ_{li} with finite mean μ_{L_i} and standard deviation σ_{L_i} .
3. The distribution function $H(x_i)$ of x_i belong to the class of distribution function τ_{xi} with finite mean μ_{x_i} and standard deviation σ_{x_i} .

Lemma 1: Callego and Moon (1993)

$$E(D - Q)^+ \leq \frac{(\sigma^2 + (Q - \mu)^2)^{1/2} - (Q - \mu)}{2}, \quad (12)$$

where Q is overcapacity and D is random variable with mean μ and standard deviation σ . Using the above lemma for any $H(x_i) \in \tau_{xi}$, it can be deduced that (see Appendix B):

$$E(x_i - R_i)^+ \leq \frac{1}{2} \left[\sqrt{\text{Var}(x_i) + (R_i - E(x_i))^2} - (R_i - E(x_i)) \right] = \frac{1}{2} \left(\sqrt{1 + k_i^2} - k_i \right) \sigma_{x_i} \quad (13)$$

Then, with considering the definition of β and the above inequality, Eq. (13), we have:

$$\beta_i = \frac{1}{1 + \theta_i E(x_i - R_i)^+} \quad (14)$$

$$\beta_i \geq \frac{1}{1 + \frac{\theta_i}{2} \left(\sqrt{1 + k_i^2} - k_i \right) \sigma_{x_i}} \quad (15)$$

$$\text{We assume that } \sqrt{1 + k_i^2} - k_i = \rho(k_i). \quad (16)$$

If the purchasing cost of i -th item is paid at the time of receiving order then the problem can be formulated by objective function with the random working capital and random space constraints, which are given below (see Appendix B):

$$\min \{ \text{MaximizTECU}(Q_i, k_i) \} \quad H(x_i) \in \tau_{xi}$$

subject to

$$\begin{aligned} \sum_{i=1}^n c_i [Q_i + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)] &\leq B, \\ \sum_{i=1}^n f_i [Q_i + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)] &\leq F, \end{aligned} \quad (17)$$

where maximizing TECU yields,

$$\begin{aligned} \max \quad TEUC(Q_i, k_i) &= \sum_{i=1}^n h_i \left(\frac{Q_i}{2} + k_i \sigma_{x_i} \right) + h_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2} \right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) + \\ &\frac{E(D_i)}{Q_i} \left\{ A_i + \left[\left(\pi_{i1} \times \frac{\rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right)}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) + \left(\pi_{i2} \times \frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2} \right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) \right] \right\} \end{aligned} \quad (18)$$

This problem can be solved with using several methods. We use chance-constrained programming technique in this paper, which is explained in proposition 1:

Proposition 1: (*chance-constrained*) as the name indicates, the chance-constrained programming technique can be used to solve problems involving chance constrained, i.e. constraints having finite probability of being violated, this technique originally developed by Charnes and Cooper (1959). if φ and γ are the probabilities of non-violation of the constraints then the constraints can be written as:

$$P \left[\sum_{i=1}^n f_i [Q_i + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+] \leq F \right] \geq \varphi, \quad 0 \leq \varphi \leq 1 \quad (19)$$

$$P \left[\sum_{i=1}^n c_i [Q_i + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+] \leq B \right] \geq \gamma, \quad 0 \leq \gamma \leq 1 \quad (20)$$

First, we use chance-constrained technique for space constraint:

$$\varphi \leq P \left\{ \sum_{i=1}^n f_i E(x_i) - \sum_{i=1}^n f_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2} \right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) + F \geq \sum_{i=1}^n f_i (Q_i + R_i) \right\} \quad (21)$$

Thus, with Markov inequality formulation above Eq. (21) is changed as follow:

$$P(x \geq a) \leq \frac{E(x)}{a} \quad (\text{Markov inequality}) \quad (22)$$

$$\varphi \leq E \left[\sum_{i=1}^n f_i E(x_i) - \sum_{i=1}^n f_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2} \right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) + F \right] / \sum_{i=1}^n f_i (Q_i + R_i)$$

Or

$$\varphi \sum_{i=1}^n f_i (Q_i + R_i) \leq \sum_{i=1}^n f_i E(x_i) - \sum_{i=1}^n f_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2} \right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) + F \quad (23)$$

By the similar way working capital constraint is changed as follow:

$$\gamma \sum_{i=1}^n c_i(Q_i + R_i) \leq \sum_{i=1}^n c_i E(x_i) - \sum_{i=1}^n c_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) + B \quad (24)$$

Therefore, our model is reduced to:

$$\min \{ \max \text{TECU}(Q_i, k_i) \}$$

$$\max \text{TECU}(Q_i, k_i) = \sum_{i=1}^n h_i \left(\frac{Q_i}{2} + k_i \sigma_{x_i} \right) + h_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) +$$

$$\frac{E(D_i)}{Q_i} \left\{ A_i + \left[\left(\pi_{i1} \times \frac{\rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right)}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) + \left(\pi_{i2} \times \frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) \right] \right\}$$

subject to

$$\begin{aligned} \gamma \sum_{i=1}^n c_i(Q_i + E(x_i) + k_i \sigma_{x_i}) + \sum_{i=1}^n c_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) - \sum_{i=1}^n c_i E(x_i) - B &\leq 0 \\ \varphi \sum_{i=1}^n f_i(Q_i + E(x_i) + k_i \sigma_{x_i}) + \sum_{i=1}^n f_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) - \sum_{i=1}^n f_i E(x_i) - F &\leq 0 \end{aligned} \quad (25)$$

We can solve this model with Lagrange multiplier method. Therefore, the Lagrange function will be:

$$\begin{aligned} \text{TECU}(Q_i, k_i, \lambda_1, \lambda_2) &= \text{TECU}(Q_i, k_i) + \\ \lambda_1 &\left[\gamma \sum_{i=1}^n c_i(Q_i + E(x_i) + k_i \sigma_{x_i}) + \sum_{i=1}^n c_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) - \sum_{i=1}^n c_i E(x_i) - B \right] + \\ \lambda_2 &\left[\varphi \sum_{i=1}^n f_i(Q_i + E(x_i) + k_i \sigma_{x_i}) + \sum_{i=1}^n f_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) - \sum_{i=1}^n f_i E(x_i) - F \right] \end{aligned} \quad (26)$$

To minimize the above unconstrained function, the Kuhn-Tucker condition for the minimization of a function subject to two inequality constraints are invoked as follow:

$$\frac{\partial \text{TECU}(Q_i, k_i, \lambda_1, \lambda_2)}{\partial Q_i} = 0 \quad (i = 1 \dots n) \quad (27)$$

$$\frac{\partial \text{TECU}(Q_i, k_i, \lambda_1, \lambda_2)}{\partial k_i} = 0 \quad (i = 1 \dots n) \quad (28)$$

$$\lambda_1 \left[\gamma \sum_{i=1}^n c_i(Q_i + E(x_i) + k_i\sigma_{x_i}) + \sum_{i=1}^n c_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) - \sum_{i=1}^n c_i E(x_i) - B \right] = 0 \tag{29}$$

$$\lambda_2 \left[\varphi \sum_{i=1}^n f_i(Q_i + E(x_i) + k_i\sigma_{x_i}) + \sum_{i=1}^n f_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) - \sum_{i=1}^n f_i E(x_i) - F \right] = 0 \tag{30}$$

By simultaneously solving the above equations for $Q_i, k_i, \lambda_1, \lambda_2$ ($i = 1, \dots, n$) the minimum point $TECU(Q_i, k_i)$ is obtained. Partial derivatives are obtained as follow:

$$\frac{\partial TECU(Q_i, k_i, \lambda_1, \lambda_2)}{\partial Q_i} = \frac{h_i}{2} - \frac{E(D_i)}{Q_i^2} \left\{ A_i + \left[\left(\pi_{i1} \times \frac{\rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right)}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) + \left(\pi_{i2} \times \frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) \right] + \lambda_1 \gamma c_i + \lambda_2 \varphi f_i \right\} = 0 \tag{27}$$

From partial derivative Eq. (27), Q_i is obtained as follow:

$$Q_i = \left[\frac{E(D_i) \left\{ A_i + \left[\left(\pi_{i1} \times \frac{\rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right)}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) + \left(\pi_{i2} \times \frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i} \right) \right] \right\}}{\frac{h_i}{2} + \lambda_1 \gamma c_i + \lambda_2 \varphi f_i} \right]^{\frac{1}{2}} \tag{31}$$

$$\frac{\partial TECU(Q_i, k_i, \lambda_1, \lambda_2)}{\partial k_i} = h_i \sigma_{x_i} + h_i \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i \tau(k_i) + \frac{E(D_i)}{Q_i} \left[\left(\frac{\sigma_{x_i}}{2}\right) \delta(k_i) \pi_{i1} + \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i \tau(k_i) \pi_{i2} \right] + \lambda_1 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i \tau(k_i) c_i + \lambda_2 \left(\frac{\sigma_{x_i}}{2}\right)^2 \theta_i \tau(k_i) f_i + \lambda_1 \gamma c_i \sigma_{x_i} + \lambda_2 \varphi f_i \sigma_{x_i} = 0$$

where

$$\tau(k_i) = \frac{2 \left(\frac{k_i}{\sqrt{1+k_i^2}} - 1 \right) \rho(k_i) \left(1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i \right) - \theta_i \left(\frac{k_i}{\sqrt{1+k_i^2}} - 1 \right) \left(\frac{\sigma_{x_i}}{2}\right) (\rho(k_i))^2}{\left(1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2}\right) \theta_i \right)^2} \tag{28}$$

$$\delta(k_i) = \frac{\left(\frac{k_i}{\sqrt{1+k_i^2}} - 1 \right) \left(1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i \right) - \theta_i \left(\frac{k_i}{\sqrt{1+k_i^2}} - 1 \right) \left(\frac{\sigma_{x_i}}{2} \right) \rho(k_i)}{\left(1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i \right)^2}$$

β_i^* is obtained as follow:

$$\beta_i^* = \frac{1}{1 + \frac{\theta_i}{2} \left(\sqrt{1+k_i^2} - k_i \right) \sigma_{x_i}} \quad (i = 1 \dots n) \quad (32)$$

The solution procedure is as follow:

Step1 obtain from partial derivative of Eq. (27), Q_i in Eq. (31). Put $Q_i (i = 1 \dots n)$ in Eq. (28), Eq. (29) and Eq. (30).

Step2 obtain λ_1, λ_2 and $k_i (i = 1 \dots n)$ by the simultaneously solving Eqs (28), (29), (30).

Step3 put λ_1, λ_2 and $k_i (i = 1 \dots n)$ in (31) and find $Q_i (i = 1 \dots n)$.

Step4 put $k_i (i = 1 \dots n)$ in Eq. (32) and find $\beta_i (1 \dots n)$.

Step5 put $k_i (i = 1 \dots n)$ and $Q_i (i = 1 \dots n)$ and $\beta_i (i = 1 \dots n)$ in Eq. (26) and find TECU.

4. Numerical examples

Example1: To illustrate the developed model, consider the numerical data, which has been stated in Table 1 (two items have been considered). Table 2 shows the distribution functions of monthly demand and lead time in days. These two distribution functions are independent. The maximum inventory investment is $B = 15000$ \$ and the total space is $F = 13000 M^2$. We consider $\gamma = 0.9$ and $\varphi = 0.9$.

Table 1

The model's parameter

Product	hi per unit per month	ci per unit	fi per unit	π_1 per unit	π_2 per unit	Ai per cycle
1	10	55	65	10	12	45
2	3.5	77	50	8.5	10	54

Table 2

The distribution functions of monthly demand and lead time in day

Monthly demand D1	Probability f(d1)	Lead time L1	Probability f(l1)	Monthly demand D2	Probability f(d2)	Lead time L2	Probability f(l2)
385	0.06	4	0.1	530	0.09	1	0.1
393	0.15	5	0.23	540	0.15	2	0.15
398	0.31	6	0.35	549	0.26	3	0.50
405	0.35	7	0.30	554	0.30	4	0.25
412	0.13	8	0.02	557	0.20		

Mean of distribution demand during lead time is:

$$E(D_1) \times E(L_1) = E(x_1) = 400 \times \frac{6}{30} = 80 \text{ units per month}$$

$$E(D_2) \times E(L_2) = E(x_2) = 55 \text{ units per month}$$

Variance of demand during lead time is:

$$Var(x_1) = Var(D_1) \times E(L_1) + (E(D_1))^2 \times Var(L_1) = 169 \text{ units per month}$$

$$Var(x_2) = Var(D_2) \times E(L_2) + (E(D_2))^2 \times Var(L_2) = 324 \text{ units per month}$$

In the Table below, we consider different values for backorder parameter, which is shown the importance of shortage for i-th item. When importance of shortage for i-th item is increased, the safety factor for i-th item will be increased, consequently. It means that the safety stock and holding cost is increased and optimal ordering quantity is decreased by Eq. (31). Table 3 shows that the larger amount of backorder rate is, the smaller amount of total expected cost (TECU) would be.

Table 3
Optimal value in terms of various backorder parameters

	k_i^*	Q_i^*	β_i^*	λ	TECU*
$\theta_1 = 0$	$k_1 = 0.64$	$Q_1 = 74$	$\beta_1 = 1$	$\lambda_1 = 0.0074$	
$\theta_2 = 0$	$k_2 = 1.29$	$Q_2 = 143$	$\beta_2 = 1$	$\lambda_2 = 0.00$	1516
$\theta_1 = 0.25$	$k_1 = 0.84$	$Q_1 = 72$	$\beta_1 = 0.53$	$\lambda_1 = 0.0099$	
$\theta_2 = 0.25$	$k_2 = 1.46$	$Q_2 = 137$	$\beta_2 = 0.56$	$\lambda_2 = 0.00$	1558
$\theta_1 = 0.5$	$k_1 = 0.88$	$Q_1 = 72$	$\beta_1 = 0.37$	$\lambda_1 = 0.0109$	
$\theta_2 = 0.5$	$k_2 = 1.49$	$Q_2 = 136$	$\beta_2 = 0.42$	$\lambda_2 = 0.00$	1574
$\theta_1 = 1$	$k_1 = 0.91$	$Q_1 = 72$	$\beta_1 = 0.23$	$\lambda_1 = 0.0119$	
$\theta_2 = 1$	$k_2 = 1.51$	$Q_2 = 134$	$\beta_2 = 0.26$	$\lambda_2 = 0.00$	1587
$\theta_1 = 5$	$k_1 = 0.92$	$Q_1 = 72$	$\beta_1 = 0.05$	$\lambda_1 = 0.0128$	
$\theta_2 = 5$	$k_2 = 1.52$	$Q_2 = 133$	$\beta_2 = 0.07$	$\lambda_2 = 0.00$	1605
$\theta_1 = 10$	$k_1 = 0.92$	$Q_1 = 72$	$\beta_1 = 0.02$	$\lambda_1 = 0.0131$	
$\theta_2 = 10$	$k_2 = 1.52$	$Q_2 = 133$	$\beta_2 = 0.04$	$\lambda_2 = 0.00$	1607
$\theta_1 = 100$	$k_1 = 0.92$	$Q_1 = 72$	$\beta_1 = 0.003$	$\lambda_1 = 0.0131$	
$\theta_2 = 100$	$k_2 = 1.52$	$Q_2 = 133$	$\beta_2 = 0.004$	$\lambda_2 = 0.00$	1610
$\theta_1 = \infty$	$k_1 = 0.92$	$Q_1 = 72$	$\beta_1 = 0$	$\lambda_1 = 0.0131$	
$\theta_2 = \infty$	$k_2 = 1.52$	$Q_2 = 133$	$\beta_2 = 0$	$\lambda_2 = 0.00$	1611

Example2: In this example, we consider four items and the problem’s inputs are briefly given by Table 4. Maximum inventory investment is $B = 15000$ \$ and total space is $F = 13000M^2$.

Table 4
Problem’s input for numerical example 2

Products	hi per unit per month	ci per unit	fi per unit	π_1 per unit	π_2 per unit	Ai per cycle
1	10	55	65	10	12	45
2	3.5	77	50	8.5	10	54
3	6	80	45	11.5	19	45
4	8	70	55	10	17	54

Other parameters are:

$$\sigma_{x_1} = 13, \sigma_{x_2} = 18, \sigma_{x_3} = 19, \sigma_{x_4} = 17, \gamma = 0.9, \varphi = 0.9, \theta_i = 1 (i=1, 2, 3, 4)$$

$$E(D_1) = 400, E(D_2) = 550, E(D_3) = 600, E(D_4) = 380$$

Final solution is

$$Q_1^* = 38, k_1^* = 0.76, Q_2^* = 48, k_2^* = 0.96$$

$$Q_3^* = 49, k_3^* = 1.24, Q_4^* = 40, k_4^* = 0.67$$

$$TECU^* = 5121 \text{ \& } \lambda_1 = 0.3033, \lambda_2 = 0$$

$$\beta_1^* = 0.21 \text{ \& } \beta_2^* = 0.20 \text{ \& } \beta_3^* = 0.23 \text{ \& } \beta_4^* = 0.18$$

Table 5

changes in $(Q_i, k_i, \beta_i, TECU)$ in terms of changes in maximum working capital and total space

	k_i^*	Q_i^*	β_i^*	λ	TECU*
Total space = 13000	$k_1 = 0.77$	$Q_1 = 39$	$\beta_1 = 0.21$		
Total capital = 19000	$k_2 = 1.15$	$Q_2 = 63$	$\beta_2 = 0.22$	$\lambda_1 = 0.00$	4393
	$k_3 = 1.46$	$Q_3 = 66$	$\beta_3 = 0.25$	$\lambda_2 = 0.240$	
	$k_4 = 0.75$	$Q_4 = 47$	$\beta_4 = 0.19$		
Total space = 10000	$k_1 = 0.63$	$Q_1 = 30$	$\beta_1 = 0.19$		
Total capital = 15000	$k_2 = 0.93$	$Q_2 = 46$	$\beta_2 = 0.20$	$\lambda_1 = 0.00$	5461
	$k_3 = 1.26$	$Q_3 = 50$	$\beta_3 = 0.23$	$\lambda_2 = 0.510$	
	$k_4 = 0.61$	$Q_4 = 36$	$\beta_4 = 0.17$		
Total space = 15000	$k_1 = 0.85$	$Q_1 = 51$	$\beta_1 = 0.22$		
Total capital = 20000	$k_2 = 1.21$	$Q_2 = 69$	$\beta_2 = 0.23$	$\lambda_1 = 0.126$	4120
	$k_3 = 1.48$	$Q_3 = 68$	$\beta_3 = 0.25$	$\lambda_2 = 0.00$	
	$k_4 = 0.80$	$Q_4 = 54$	$\beta_4 = 0.19$		

We consider three different values for the maximum inventory investment and maximum space and obtain Q_i, k_i, β_i and $TECU$ based on these different values. We tabulate final solution in Table 5. According to Table 5, the bigger total space and total inventory investment is, the smaller $TECU$ (total expected cost per unit time) would be.

5. Conclusion

The purpose of this study was to extend multi product inventory system by adding two limitations (working capital and space) and considering backorder rate as a decision variable which is dependent on the expected demand during shortage. It means that when shortage happen, the larger amount of shortage is, the smaller ratio of client can wait and therefore, the smaller backorder would be. We supposed that distribution of demand during lead time was unknown, but the mean and variance of demand during lead time was known. With this assumption, we utilized minimax distribution free method to minimize our model. With chance programming method, we transformed our random

constraints to the crisp constraints. Then, we applied Lagrange multiplier method to solve our model. At the end of paper, we prepared two numerical examples and compared them together.

Appendix A:

The reorder point is:

$$R_i = E(x_i) + k_i \sigma_{x_i} \rightarrow k_i \sigma_{x_i} = R_i - E(x_i) \quad (\text{A.1})$$

Safety stock is:

$$SS_i = R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+ \quad (\text{A.2})$$

The expected inventory during each cycle is:

$$\bar{I} = \frac{Q_i}{2} + SS_i \rightarrow \bar{I} = \frac{Q_i}{2} + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+ \quad (\text{A.3})$$

The expected cycle length for i-th item is:

$$\frac{Q_i}{E(D_i)} \quad (\text{A.4})$$

Therefore, the expected holding cost per cycle is as follow:

$$\begin{aligned} \frac{Q_i}{E(D_i)} \left[\frac{Q_i}{2} + R_i - E(x_i) + (1 - \beta_i)E(x_i - R_i)^+ \right] \times h_i = \\ \frac{Q_i}{E(D_i)} \left[\frac{Q_i}{2} + k_i \sigma_{x_i} + (1 - \beta_i)E(x_i - R_i)^+ \right] \times h_i \end{aligned} \quad (\text{A.5})$$

Appendix B

Since

$$(x_i - R_i)^+ = \frac{|x_i - R_i| + (x_i - R_i)}{2} \quad (\text{B.1})$$

The result follows by taking expectations and by using the Cauchy-Schwarz Inequality:

$$\begin{aligned} E|x_i - R_i| &\leq [E(x_i - R_i)^2]^{\frac{1}{2}} = [E\{(x_i - E(x_i)) - (R_i - E(x_i))\}^2]^{\frac{1}{2}} \\ &= [E\{(x_i - E(x_i))^2 - 2(x_i - E(x_i))(R_i - E(x_i)) + (R_i - E(x_i))^2\}]^{\frac{1}{2}} \end{aligned} \quad (\text{B.2})$$

Therefore:

$$E(x_i - R_i)^+ = \frac{E|x_i - R_i| + E(x_i - R_i)}{2} \leq \frac{\sqrt{\sigma_{x_i}^2 + (R_i - E(x_i))^2} - (R_i - E(x_i))}{2} =$$

$$\frac{1}{2} \left(\sqrt{1 + k_i^2} - k_i \right) \sqrt{\text{Var}(D_i) \times E(L_i) + (E(D_i))^2 \times \text{Var}(L_i)} = \frac{1}{2} \left(\sqrt{1 + k_i^2} - k_i \right) \sigma_{x_i} \quad (\text{B.3})$$

Because:

$$R_i = E(x_i) + k_i \sigma_{x_i} \quad (\text{B.4})$$

Total expected cost per unit time is:

$$\text{TECU}(Q_i, R_i, \beta_i) = \sum_i \frac{E(D_i)}{Q_i} A_i + \left[\frac{Q_i}{2} + R_i - E(x_i) + (1 - \beta_i) E(x_i - R_i)^+ \right] \times h_i$$

$$+ \frac{E(D_i)}{Q_i} [\pi_{i1} \beta_i E(x_i - R_i)^+ + \pi_{i2} (1 - \beta_i) E(x_i - R_i)^+] \quad (\text{B.5})$$

From definition of β we have:

$$\beta_i = \frac{1}{1 + \theta_i E(x_i - R_i)^+} \quad (\text{B.6})$$

$$\beta_i \geq \frac{1}{1 + \frac{\theta_i}{2} \left(\sqrt{1 + k_i^2} - k_i \right) \sigma_{x_i}} \quad (\text{B.7})$$

With the definition of β and equation (A.4) total expected cost per unit time is changed as follow:

(B.8)

$$\text{TECU}(Q_i, k_i) \leq$$

$$\sum_{i=1}^n h_i \left(\frac{Q_i}{2} + k_i \sigma_{x_i} \right) + h_i \left[\left(1 - \frac{1}{1 + \left(\frac{\sigma_{x_i}}{2} \right) \left(\sqrt{1 + k_i^2} - k_i \right) \theta_i} \right) \left(\sqrt{1 + k_i^2} - k_i \right) \left(\frac{\sigma_{x_i}}{2} \right) \right] +$$

$$\frac{E(D_i)}{Q_i} \left\{ A_i + \left[\left(\pi_{i1} \times \frac{1}{1 + \left(\sqrt{1 + k_i^2} - k_i \right) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) \left(\sqrt{1 + k_i^2} - k_i \right) \left(\frac{\sigma_{x_i}}{2} \right) + \right. \right.$$

$$\left. \left. \left(1 - \frac{1}{1 + \left(\frac{\sigma_{x_i}}{2} \right) \left(\sqrt{1 + k_i^2} - k_i \right) \theta_i} \right) \left(\sqrt{1 + k_i^2} - k_i \right) \left(\frac{\sigma_{x_i}}{2} \right) \right] \right\}$$

We assume that $\sqrt{1 + k_i^2} - k_i = \rho(k_i)$

Therefore, our model is reduced to:

$$\max \quad TECU(Q_i, k_i) = \sum_{i=1}^n h_i \left(\frac{Q_i}{2} + k_i \sigma_{x_i} \right) + h_i \left(\frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2} \right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) +$$

$$\frac{E(D_i)}{Q_i} \left\{ A_i + \left[\left(\pi_{i1} \times \frac{\rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right)}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) + \left(\pi_{i2} \times \frac{\rho(k_i)^2 \left(\frac{\sigma_{x_i}}{2} \right)^2 \theta_i}{1 + \rho(k_i) \left(\frac{\sigma_{x_i}}{2} \right) \theta_i} \right) \right] \right\} \quad (\text{B.9})$$

Acknowledgment

The authors would like to thank the anonymous referees for their constructive comments on earlier version of this paper.

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