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An operating cost minimization model for buyer-vendor coordination batch system with breakdowns, scrap, overtime, and an external source

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C H R O N I C L E

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A B S T R A C T

When making a batch production decision for a buyer-vendor coordination system, the management must simultaneously consider the operating expenses incurred in in-house manufacturing and inventory, finished goods' shipping, and stock holding at the retailer end. Achieving the operational goals of desirable quality, minimal production disruption, and shortening fabrication time help minimize overall in-house operating costs and maximize customer satisfaction. This work builds an operating cost minimization model for buyer-vendor coordination batch system with scrap, breakdowns, overtime, multi-shipment, and an external source to assist the management in optimizing their production-delivery plan. Removing inevitable scrap items ensures product quality, and correction action on stochastic equipment breakdown prevents unacceptable production delays. Implementing partial overtime and adopting an external source expedites in-house manufacturing time. Model construction and cost analysis enable us to decide the operating expense function. Then, we verify the function's convexity and decide our model's best manufacturing runtime with the differential calculus and a proposed algorithm. Furthermore, the numerical demonstrations are used to exhibit our work's applicability and show what kinds of crucial in-depth information can be disclosed and made accessible to the production planners for their decision-making.

1. Introduction

This study develops an operating cost minimization model for buyer-vendor coordination batch system with scrap items, breakdowns, overtime, multi-shipment, and an external source to assist the management of manufacturers in optimizing a competitive production-delivery coordinated plan. Specifically, integrating production and shipping/sale units in global firms to minimize the total operating expenses is crucial to managers' planning and operations. Ben-Daya and Al-Nassar (2008) studied an integrated inventory-production model in a supply chain (SC) comprising three coordinated layers: retailers, producers, and suppliers. Multiple equal-size shipments are delivered to the next layer in the SC, disregarding the production completion of the entire lot. Also, the cycle time at each layer is a multiple of its downstream layer's cycle time. Finally, the researchers constructed a cost-minimization model to determine the optimal coordinated inventory-production policy and presented numerical illustrations for the solution procedure and significant cost savings. Mawandiya et al. (2017) explored a production and inventory closed-loop supply chain featuring the finite fabricating and refabricating rates. Their supply chain comprised a manufacturer, a retailer, and a remanufacturer. The retailer uses either new or refabricated goods received to meet customers' steady demand. The manufacturer produces goods in batches using outside suppliers' raw materials. The remanufacturer refabricates using used items steadily offered by the customers. Their study aimed to derive a production-inventory decision to minimize the combined operating expenses of all parties in the studied supply chain. The researchers provided a numerical illustration to show their solution procedure and the sensitivity analysis to expose the binding parameters'

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effect on the problem and comparison with the existing works. Khorshidvand et al. (2021) presented a method to make supply chain (SC) coordination decisions involving closed-loop SC network design objectives to maximize profit and minimize CO2 emission. Their nonlinear programming model aimed to create an advertisement, greenness, and price decisions to meet customers' indeterminate demands and return rate tolerance to the green quality. The researchers resolve large-scale examples in a reasonable running time through robust optimization modeling and the Lagrangian relaxing algorithm. Lastly, they concluded that (i) appropriate green and advertising policies improve the environmental and profitable goals, and (ii) their model's obtained result outperforms the same model without considering the robust optimization method. Extra works (Goyal & Gupta, 1989; Hoque & Goyal, 2006; Sarmah et al., 2006; Tseng et al., 2014; Zhang et al., 2016; Abushaega et al., 2021; Çömez-Dolgan et al., 2021; Khan et al., 2021; Perarasi et al., 2021; Pichka et al., 2022; Karimpoor et al., 2023) discovered the influence of various product-shipping policies on optimizing the global firms' internal or coordinated supply-chain systems.

In addition to coordinating the fabrication and shipping units, the management of global firms must meet the shorter trend in customers' order lead times. Inspired by achieving this goal, the proposed study incorporates an in-house overtime strategy and an external provider in our model to effectively shorten fabricating uptime to rapidly satisfy customers' short lead-time needs. Morita (2010) analyzed the dynamics of outsourcing. The study built a decision model to explore whether to outsource the intermediate products from the North to the firms in the South. The final goods are made in the North, and the subcontracting price in the North for the intermediate products is high, but the fixed outsourcing cost is low. Conversely, if they outsource the intermediate products to the South, the variable cost is low, but the fixed outsourcing cost is high. The North-South endogenous growth's steady states and their transition paths are analyzed to explore the dynamics of outsourcing and decision-making influenced by the wage and economic development between these regions. Conway and Sturges (2014) examined the relationship between the contracted work hours and unpaid overtime work (including part- and full-time work) in Britain. The study tested the hypotheses regarding contracted work and unpaid overtime using a British data set derived from 735 workplaces involving 4530 unpaid worker samples. In addition, they examined unpaid overtime for different genders, occupations, and work arrangement flexibility. They found that (i) for full- and part-time: more unpaid overtime hours existed among part-time workers than full-time workers. (ii) for gender: men worked unpaid overtime more than women. (iii) for occupation: professional/managerial part-time workers work unpaid overtime more than others. Barak and Javanmard (2020) applied the fuzzy sets for exploring an outsourcing model to select and build strategic partners and alliances efficiently. The researchers proposed an interval-based two-phase scheme to cope with the problem above. The researchers initially found and weighted the effective alliance evaluation strategies by integrating (i) the quantitative strategic planning matrix with gap analysis and (ii) the interval-value fuzzy version of S.W.O.T. analysis (i.e., strength, weakness, opportunity, and threats). Then, they evaluated the strategic partners with four multi-criteria decision-making methods based on interval values. Lastly, they employed the utility interval method and provided sensitivity analyses to assess their results and measure their approaches' robustness with a real partner selection case from the Iranian factory. Other studies (Golden and Wiens-Tuers, 2008; Assid et al., 2015; Chiu et al., 2020; Chiu et al., 2021; Cornelius et al., 2021; Chiu et al., 2022; Porto et al., 2022; Chiu et al., 2023; Shekar and Nataraj 2023) discovered the effect of different subcontracting and overtime/output-increasing strategies on reducing batch runtime of modern manufacturing firms' operations and management.

Furthermore, the production management must retain high product quality and deal with unwanted but inevitable equipment breakdowns to meet clients' expectations and avoid delays in the manufacturing schedule. Kumar et al. (2004) studied a semiconductor fabrication rescheduling problem with the predicted/unpredicted equipment unreliability and job demand. The study comprised the following: (1) A neuro-expert Petri net model consisting of a subsystem of one-buffer and two-machine, (2) Estimating the equipment breakdown and repair rates, (3) A specific rescheduling algorithm coping with the breakdown and repair rates, and (4) Rescheduling algorithm execution. Lastly, the researchers used an example to illustrate their solution method and showed how well and reliable results with computational analysis. Hou (2007) developed a mathematical model to explore the impact of process quality and setup cost on the economic production quantity-based system's optimal cycle time and capital expenditure. The researcher focused on the influence of capital investing strategies in reducing setup and improving process quality. As a result, the study presented an efficient approach for simultaneously deciding the best fabrication batch time, process quality, and setup expense. Lastly, the researcher used numerical illustration to show the study's proposed approach and provided specific managerial implications. Rivera-Gómez et al. (2013) explored fabrication and quality control strategies for a single product deteriorating fabrication system featuring random overhauls and repairs. The wear of the equipment and human interventions influences the gradually worsened parts' quality. Upon machine failure, an overhaul or reparation is in action. The former action makes equipment's condition deteriorate following overhaul, while the latter brings the equipment back to a good as new condition. The study developed a model with the fabrication rate and switching overhaul/reparation action as its decision variables to minimize the anticipated costs involving stock holding, backlogging, and overhaul/reparation coping with deterioration. The study specifically considered the equipment repairing history regarding its operation states and historical reparation records and provided a numerical illustration of how their control strategies work. Karakatsoulis and Skouri (2021) simultaneously derived the optimal reorder point and batch size (r, Q) policy for a deterministic inventory system with full inspection implemented for random defective products and allowable shortages. To avoid unexpected shortages, the researchers set the total demand as the reorder point in inspection time and included the backlogging expenses in formulating their model. Using an optimization approach, the researchers theoretically decided an optimal (r, Q) policy that minimized overall production expenditures. Lastly, using numerical illustration and simulation, the researchers showed their model's significant cost savings and shortage reduction compared to existing models. Other works (Dohi et al., 1998; Papachristos and Konstantaras, 2006; Salehi et al., 2016; Zhang et al., 2016; Yera et al., 2021; Bozhanova et al., 2022; Chiu et al., 2022; Hashemi et al., 2022; Das et al., 2023) studied the impacts of various production defects and equipment unreliability on production systems' optimization, operations, and planning. Little works have examined the operating cost minimization model for buyer-vendor coordination batch system with scrap, breakdowns, overtime, and an external source. This work intends to fill the gap.

2. Problem description, assumption, and modelling

This work builds an operating cost minimization model for buyer-vendor coordination batch system with scrap, breakdowns, overtime, multi-shipment, and an external source. The definition of notation is as follows:

t_{1Z} = the system uptime (the decision parameter),
λ = annual product requirements,
Q = the batch size,
P_{1A} = annual fabricating rate with overtime implementing,
P_1 = standard annual production rate (i.e., without overtime implementation),
K_A = setup cost under the overtime strategy,
C_A = unit cost under the overtime strategy,
K = regular setup cost,
C = regular unit cost,
α_1 = the relating factor between overtime fabrication rate P_{1A} and standard rate P_1,
α_2 = the relating factor between K_A and K,
α_3 = the relating factor between C_A and C,
β = average annual Poisson-distributed failures,
M = the equipment repairing cost,
t = the time before a Poisson-distributed failure takes place,
t_r = the fixed machine repair time,
π = the outsourcing proportion in a batch ($0 < \pi < 1$),
K_π = fixed outsourcing cost,
C_π = unit outsourcing cost,
β_1 = the relating factor between K_π and K,
β_2 = the relating factor between C_π and C,
T'_Z = the cycle time in condition 1,
t'_{2Z} = the stock delivering time in condition 1,
t'_{nZ} = the delivering time interval in condition 1,
x = the uniform-distributed scrap rate,
d_{1A} = production rate of scrap item in t_{1Z} ($d_{1A} = xP_{1A}$),
C_S = unit scrap item's disposal cost,
h = unit holding cost,
h_2 = unit holding cost at buyer side,
C_1 = safety stock's unit cost,
h_3 = safety stock's unit holding cost,
K_1 = fixed delivery cost,
C_T = unit delivering cost,
n = number of shipments per cycle,
g = t_r, the fixed machine repair time,
H_0 = end product's status when a failure happens,
H_1 = end product's position when t_{1Z} ends,
H = end product's status after outsourced stocks are received,
T_Z = cycle time in condition 2,
t_{2Z} = rework time in condition 2,
t_{3Z} = stock delivering time in condition 2,
t_{nZ} = the delivering time interval in condition 2,
T = system's cycle time without overtime, machine failure, nor outsourcing,
t_1 = uptime for a system without overtime, machine failure, nor outsourcing,
t_2 = rework time for a system without overtime, machine failure, nor outsourcing,
t_3 = stock delivering time for a system without overtime, machine failure, nor outsourcing,
d_1 = scrap stock's production rate for a system without overtime, machine failure, nor outsourcing,
I = the leftover products at the end of each delivering interval,
D = the quantity per delivery,
$TC(t_{1Z})_1$ = condition 1's total cycle cost,
$E[TC(t_{1Z})_1]$ = condition 1's expected total cycle cost,

$E[T'_Z]$ = condition 1's expected cycle time,
$I(t)$ = end product's level at time t,
$I_F(t)$ = safety stock's level at time t,
$TC(t_{1Z})_2$ = condition 2's total cycle cost,
$E[TC(t_{1Z})_2]$ = condition 2's expected total cycle cost,
$E[T_Z]$ = condition 2's expected cycle time,
$I_s(t)$ = scrap stock's level at time t,
$I_c(t)$ = buyer stock's level at time t,
$\boldsymbol{T_Z}$ = the replenishing cycle time of this study,
$E[TCU(t_{1Z})]$ = this expected system cost.

Suppose a batch fabrication plan in the buyer-vendor coordination environment must satisfy the annual product requirements λ. In each Q, a π portion is provided by an outside contractor to cut short the needed uptime. The proposed model also implements an overtime strategy with output rate P_{1A} to fabricate the other $(1-\pi)$ portion of batch size (where P_{1A} is α_1 more rapidly than the standard fabricating rate P_1 (see Eq. (1)).

$$P_{1A} = (1+\alpha_1)P_1 \tag{1}$$

The consequent cost-relating parameters versus standard cost variables (such as C_A, in-house setup K_A, unit purchase cost C_π, and fixed setup cost K_π) are assumed below (see to Eqs. (2) to (5)):

$$C_A = (1+\alpha_3)C \tag{2}$$

$$K_A = (1+\alpha_2)K \tag{3}$$

$$C_\pi = (1+\beta_2)C \tag{4}$$

$$K_\pi = (1+\beta_1)K \tag{5}$$

The in-house production equipment is not reliable. It may produce an x proportion of scrap products randomly. We screen and identify scraps and dispose of them at the cost of C_S per item. Meanwhile, the external supplier promises the quality and delivery schedule (i.e., outsourced items arrive when in-house uptime ends). Furthermore, the in-house machine is subject to the Poisson-distributed failures with the mean β instances per year. A machine failure may or may not occur during the uptime t_{1Z}. Hence, this study must explicitly investigate the following two conditions:

2.1. Condition one: A failure occurs during fabrication uptime

Since a failure occurs during uptime, $t < t_{1Z}$, we adopt an abort/resume (AR) inventory policy to handle the machine failure instance. A machine-repair task is initiated right away, and the interrupted/undone batch resumes fabrication when the machine is stored. Figure 1 shows our batch fabrication system's inventory status featuring external source, overtime, stochastic failures, and scrap in a buyer-vendor coordination environment. It indicates that when a random failure occurs, the inventory level reaches H_0, and it remains at the level of H_0 until the machine is fixed/restored at the end of repair time t_r. Here, we assume a constant repair time t_r; if the time required to correct the failure exceeds t_r, a spare/rental machine will be in place to avoid unwanted fabrication delays. The stock level piles up H_1 when t_{1Z} ends. Then, upon receipt of outsourcing stocks, the inventory level rises to H. Finally, the delivery of n equal-size shipments in each fixed time interval t'_{nZ} gradually depletes the finished batch during t'_{2Z} (see Fig. 1 for details). Since the stock-out situation is not allowed, we must have $(P_{1A} - d_{1A} - \lambda) > 0$.

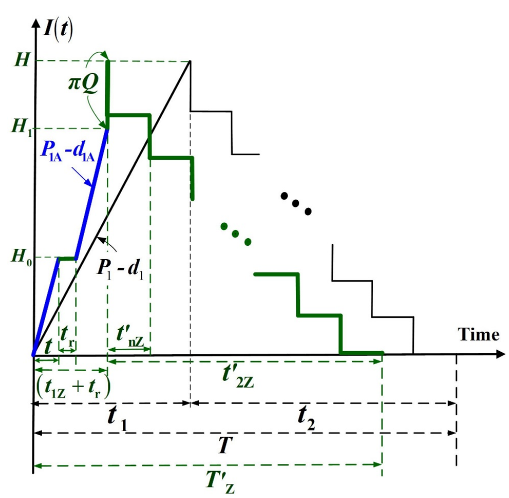

Fig. 1. The inventory status for the proposed buyer-vendor coordination batch system with scrap, breakdowns, overtime, and an external source (in thicker line) compared to a problem with only scrapped items (see the thinner lines)

Fig. 2 illustrates the level of safety stocks in T'_Z of condition one. Since t_r is added to the cycle time T'_Z, the safety stocks must also be included in the finished batch and shipped to the client to satisfy the extra demand.

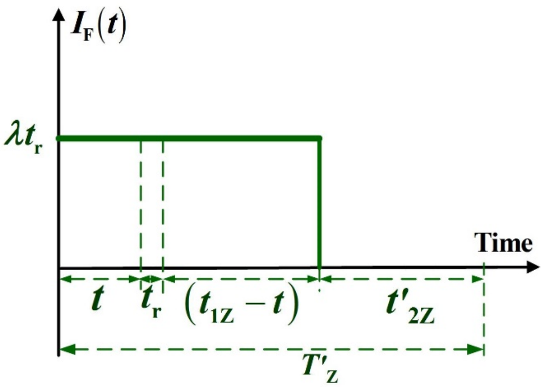

Fig. 2. The level of safety stocks in T'_Z of condition one

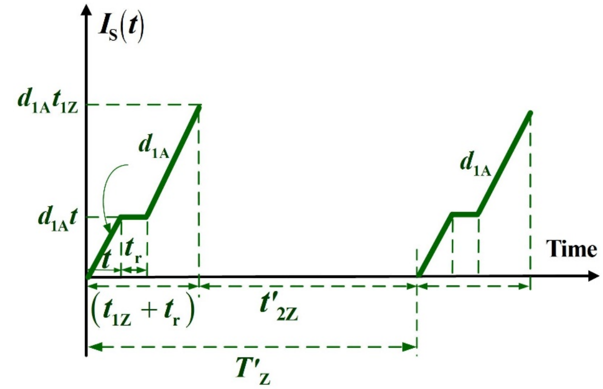

Fig. 3. The inventory level of scrap in T'_Z of condition one

Fig. 3 depicts the inventory level of scraps in T'_Z of condition one. It indicates the maximum level of scrap is at $d_{1A}t_{1Z}$.

We observe the following formulas according to our model description and Fig. 1 to Fig. 3:

$$H_0 = (P_{1A} - d_{1A})t \tag{6}$$

$$H_1 = (P_{1A} - d_{1A})t_{1Z} \tag{7}$$

$$t_{1Z} = \frac{H_1}{P_{1A} - d_{1A}} = \frac{Q(1-\pi)}{P_{1A}} \tag{8}$$

$$H = H_1 + \pi Q + \lambda t_r \tag{9}$$

$$T'_Z = t_{1Z} + t_r + t'_{2Z} \tag{10}$$

$$t'_{2Z} = T'_Z - (t_{1Z} + t_r) \tag{11}$$

The maximum level of scrap is

$$d_{1A}t_{1Z} = x\left[(1-\pi)Q\right] = xP_{1A}t_{1Z} \tag{12}$$

The total stocks during t'_{2Z} can be computed as follows,

$$\left(\frac{n-1}{2n}\right)H(t'_{2Z}) \tag{13}$$

Fig. 4 illustrates the inventory level at the buyer's side in T'_Z of condition one. The total buyer inventories in T'_Z are

$$\frac{1}{2}\left[\frac{Ht'_{2Z}}{n} + T'_Z(H - \lambda t'_{2Z})\right] \tag{14}$$

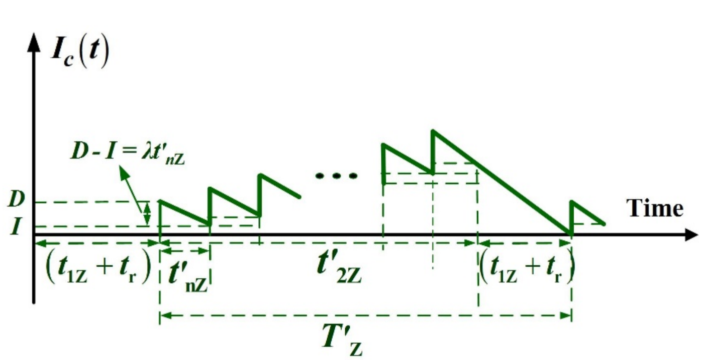

Fig. 4. The inventory level at the buyer's side in T'_Z of condition one

$TC(t_{1Z})_1$ in condition one includes: the outsourcing cost and manufacturing cost (in-house), fixed and variable delivering costs, machine repairing cost (see Fig. 1), safety stock relevant cost (see Fig. 2), disposal cost (see Fig. 3), and overall holding expenses (comprising the finished and scrap items, and buyer's inventories) during T'_Z as displayed Eq. (15).

$$\begin{aligned} TC(t_{1Z})_1 = {} & C_\pi(\pi Q) + K_\pi + (1-\pi)QC_A + K_A + nK_1 + C_T\left[\lambda t_r + Q\left(1-x(1-\pi)\right)\right] \\ & + M + h_3(t_{1Z}+t_r)(\lambda t_r) + x(1-\pi)QC_S + C_1\lambda t_r + \frac{h_2}{2}\left[\frac{Ht'_{2Z}}{n} + (H-\lambda t'_{2Z})T'_Z\right] \\ & + h\left[(t_{1Z})\frac{H_1 + d_{1A}t_{1Z}}{2} + (H_0 t_r) + (d_{1A}t)t_r + \left(\frac{n-1}{2n}\right)H(t'_{2Z})\right] \end{aligned} \tag{15}$$

Substitute formulas (1) to (5), and (12) in Eq. (15), we obtain the following $TC(t_{1Z})_1$:

$$\begin{aligned} TC(t_{1Z})_1 = {} & (1+\beta_2)C(\pi Q) + (1+\beta_1)K + (1+\alpha_2)K + (1+\alpha_3)C\left[(1-\pi)Q\right] + M + nK_1 + C_1\lambda t_r \\ & + C_T\left[Q\left(1-x(1-\pi)\right) + \lambda t_r\right] + h_3(\lambda t_r)(t_{1Z}+t_r) + \frac{h_2}{2}\left[\frac{Ht'_{2Z}}{n} + T'_Z(H-\lambda t'_{2Z})\right] \\ & + C_S x\left[(1-\pi)Q\right] + h\left[\frac{H_1 + x(1+\alpha_1)P_1 t_{1Z}}{2}(t_{1Z}) + H_0 t_r + x(1+\alpha_1)P_1(t)t_r + \left(\frac{n-1}{2n}\right)Ht'_{2Z}\right] \end{aligned} \tag{16}$$

2.2. Condition two: No failure occurs during uptime

No machine failure occurs during uptime so, $t > t_{1Z}$. Figure 5 exhibits our batch fabrication system's inventory status in condition 2. When t_{1Z} ends, the stock-level piles up to H_1. Upon receipt of the outsourced stocks, the stock-level reaches H, before t_{2Z} starts. Fig. 6 displays the level of safety stocks in T_Z of condition two. Fig. 4 also exhibits the scrap level of condition two.

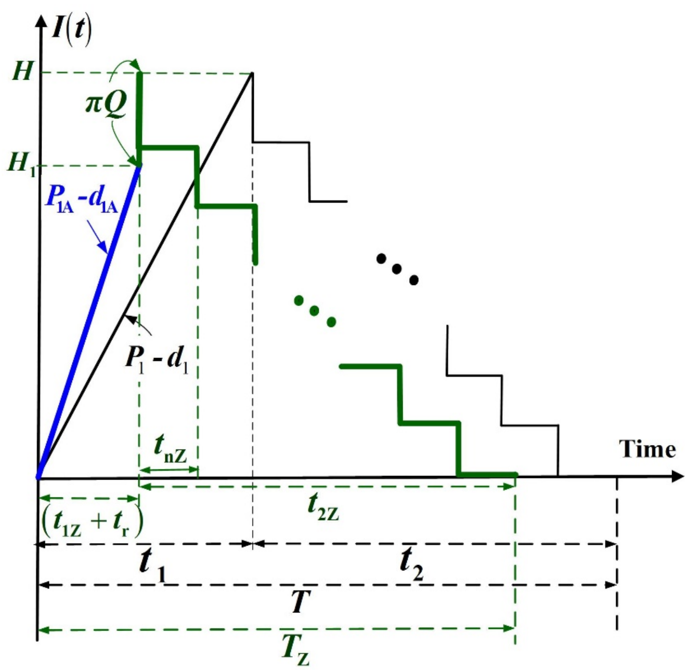

Fig. 5. The proposed system's stock-level with an outside contractor, scrap, and overtime, but with no equipment failure occurrence (see the thicker lines) than a system with only scrapped items (see the thinner lines)

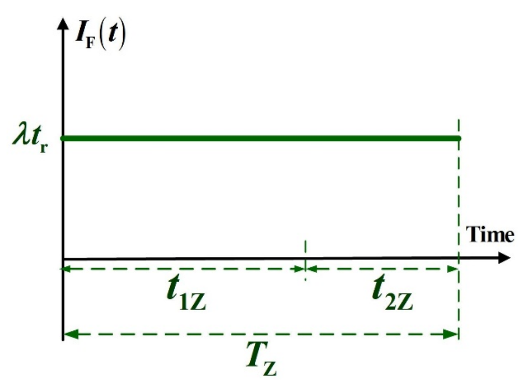

Fig. 6. The level of safety stocks in T_Z of condition two

Similarly, we observe the following formulas from condition two of the problem description and Fig. 5 to Fig. 6:

$$H_1 = (P_{1A} - d_{1A})t_{1Z} \tag{17}$$

$$t_{1Z} = \frac{H_1}{P_{1A} - d_{1A}} = \frac{(1-\pi)Q}{P_{1A}} \tag{18}$$

$$H = H_1 + \pi Q \tag{19}$$

$$T_Z = t_{1Z} + t_{2Z} \tag{20}$$

$$t_{2Z} = T_Z - t_{1Z} \tag{21}$$

The total inventories during t_{2Z} and total buyer inventories in T_Z are shown in Eq. (22) and Eq. (23), respectively (Chiu et al., 2020):

$$\left(\frac{n-1}{2n}\right) H\left(t_{2Z}\right) \tag{22}$$

$$\frac{1}{2}\left[\frac{Ht_{2Z}}{n} + T_Z\left(H - \lambda t_{2Z}\right)\right] \tag{23}$$

$TC(t_{1Z})_2$ in condition 2 includes: the outsourcing cost and manufacturing cost (in-house), delivering costs (see Figure 5), holding cost of safety stock (Fig. 6), disposal cost, and total holding costs (including scrap and finished items, and buyer's inventories) during T_Z as expressed in Eq. (24).

$$\begin{aligned} TC\left(t_{1Z}\right)_2 &= C_\pi\left(\pi Q\right) + K_\pi + (1-\pi)QC_A + K_A + nK_1 + Q\left(1 - x(1-\pi)\right)C_T \\ &+ \left(\lambda t_r\right)T_Z h_3 + x(1-\pi)QC_S + \frac{h_2}{2}\left[T_Z\left(H - \lambda t_{2Z}\right) + \frac{Ht_{2Z}}{n}\right] \\ &+ h\left[\left(\frac{n-1}{2n}\right)H\left(t_{2Z}\right) + \frac{H_1 + d_{1A}t_{1Z}}{2}\left(t_{1Z}\right)\right] \end{aligned} \tag{24}$$

Substitute formulas (1) to (5), and (12) in Eq. (24), we obtain $TC(t_{1Z})_2$ as follows:

$$\begin{aligned} TC\left(t_{1Z}\right)_2 &= \left(1+\beta_2\right)C\left(\pi Q\right) + \left(1+\beta_1\right)K + (1-\pi)Q\left(1+\alpha_3\right)C + \left(1+\alpha_2\right)K \\ &+ nK_1 + Q\left(1 - x(1-\pi)\right)C_T + h_3\left(\lambda t_r\right)T_Z + x(1-\pi)QC_S \\ &+ \frac{h_2}{2}\left[T_Z\left(H - \lambda t_{2Z}\right) + \frac{Ht_{2Z}}{n}\right] + h\left[\left(\frac{n-1}{2n}\right)Ht_{2Z} + \frac{H_1 + x\left(1+\alpha_1\right)P_1 t_{1Z}}{2}\left(t_{1Z}\right)\right] \end{aligned} \tag{25}$$

2.3. Integrating conditions 1 and 2

As we assume, the machine failure adheres to Poisson distribution with mean β instances per year. Hence, the time to failure obeys an Exponential distribution. $F(t) = (1 - e^{-\beta t})$ and $f(t) = \beta e^{-\beta t}$ and are its cumulative density and density functions. Moreover, the random scrap rate causes our cycle length to become variable. Hence, we apply the renewal reward theorem to deal with such a variable cycle time and calculate $E[TCU(t_{1Z})]$ below:

$$E\left[TCU\left(t_{1Z}\right)\right] = \frac{\left\{\int_0^{t_{1Z}} E\left[TC\left(t_{1Z}\right)_1\right]\cdot f(t)\,dt + \int_{t_{1Z}}^{\infty} E\left[TC\left(t_{1Z}\right)_2\right]\cdot f(t)\,dt\right\}}{E[\boldsymbol{T}_Z]} \tag{26}$$

where $E[\boldsymbol{T}_Z]$, $E[T'_Z]$, and $E[T_Z]$ denote the following:

$$E[\boldsymbol{T}_Z] = \int_0^{t_{1Z}} E\left[T'_Z\right]\cdot f(t)\,dt + \int_{t_{1Z}}^{\infty} E\left[T_Z\right]\cdot f(t)\,dt \tag{27}$$

$$E[T'_Z] = \frac{Q\left[1 - E[x](1-\pi)\right] + \lambda t_r}{\lambda} = \frac{t_{1Z}P_{1A}\left[\frac{1}{(1-\pi)} - E[x]\right] + \lambda t_r}{\lambda} \tag{28}$$

$$E[T_Z] = \frac{Q\left[1 - E[x](1-\pi)\right]}{\lambda} = \frac{t_{1Z}P_{1A}\left[\frac{1}{(1-\pi)} - E[x]\right]}{\lambda} \tag{29}$$

Applying $E[x]$ to Eq. (16) and Eq. (21) to deal with the randomness. Then, substitute Eq. (16), Eq. (21), and Eq. (27) in Eq. (26), with additional derivations $E[TCU(t_{1Z})]$ becomes as displayed in Eq. (30) (see Appendix A for details):

$$E\left[TCU\left(t_{1Z}\right)\right]=\left[\frac{\lambda}{\delta_1+\dfrac{\lambda g\left(1-e^{-\beta t_{1Z}}\right)}{\left(t_{1Z}\right)\left(1+\alpha_1\right)P_1}}\right]\left[\begin{array}{l}\dfrac{\delta_2}{t_{1Z}}+\delta_3+v_1\left(t_{1Z}\right)+\dfrac{W_1}{t_{1Z}}-hg\left(e^{-\beta t_{1Z}}\right)-\dfrac{W_1}{\left(t_{1Z}\right)}\left(e^{-\beta t_{1Z}}\right)\\+G_3\left(1-e^{-\beta t_{1Z}}\right)\end{array}\right] \tag{30}$$

3. Deriving the optimal fabrication uptime t_{1Z}*

By applying the first- and second-derivative of $E[TCU(t_{1Z})]$, and we obtain Eq. (A-6) and (A-7) (Appendix A). One notes that the first term on the RHS (right-hand side) of Eq. (A-7) is positive, if the second term is also positive, then $E[TCU(t_{1Z})]$ is convex. Meaning, if $y(t_{1Z}) > t_{1Z} > 0$ is true (refer to Eq. (A-8). After Eq. (A-8) is verified, to solve t_{1Z}* we can set the first-derivative of $E[TCU(t_{1Z})] = 0$ (see Eq. (A-6)). The condition becomes as follows:

$$\left\{\begin{array}{l}\left\{\left(hg+G_3\right)\left[\beta\delta_1 e^{-\beta t_{1Z}}\left(1+\alpha_1\right)P_1\right]+v_1\left[\delta_1\left(1+\alpha_1\right)P_1-\lambda g\beta e^{-\beta t_{1Z}}\right]\right\}\left(t_{1Z}\right)^2\\+\left[-\left(\delta_3-hg\right)\left(\beta\lambda g e^{-\beta t_{1Z}}\right)+2\lambda g v_1\left(1-e^{-\beta t_{1Z}}\right)+e^{-\beta t_{1Z}}W_1\beta\delta_1\left(1+\alpha_1\right)P_1\right]t_{1Z}\\-\left(\delta_2+W_1\right)\left[\delta_1\left(1+\alpha_1\right)P_1+\lambda\beta e^{-\beta t_{1Z}}g\right]-\left(\delta_3+G_3\right)\lambda g\left(e^{-\beta t_{1Z}}-1\right)\\-\left(hg+G_3\right)\lambda g\left(-e^{-2\beta t_{1Z}}+e^{-\beta t_{1Z}}\right)+e^{-\beta t_{1Z}}W_1\left[\delta_1\left(1+\alpha_1\right)P_1+\beta\lambda g\right]\end{array}\right\}=0 \tag{31}$$

Let z_0, z_1, and z_2 be the following:

$$z_0=\left[\begin{array}{l}-\left(\delta_2+W_1\right)\left[\delta_1\left(1+\alpha_1\right)P_1+\lambda\beta e^{-\beta t_{1Z}}g\right]-\left(\delta_3+G_3\right)\lambda g\left(e^{-\beta t_{1Z}}-1\right)\\-\left(hg+G_3\right)\lambda g\left(-e^{-2\beta t_{1Z}}+e^{-\beta t_{1Z}}\right)+e^{-\beta t_{1Z}}W_1\left[\delta_1\left(1+\alpha_1\right)P_1+\beta\lambda g\right]\end{array}\right]$$
$$z_1=\left[-\left(\delta_3-hg\right)\left(\beta\lambda g e^{-\beta t_{1Z}}\right)+2\lambda g v_1\left(1-e^{-\beta t_{1Z}}\right)+e^{-\beta t_{1Z}}W_1\beta\delta_1\left(1+\alpha_1\right)P_1\right]$$
$$z_2=\left(hg+G_3\right)\left[\beta\delta_1 e^{-\beta t_{1Z}}\left(1+\alpha_1\right)P_1\right]+v_1\left[\delta_1\left(1+\alpha_1\right)P_1-\lambda g\beta e^{-\beta t_{1Z}}\right]$$

We rearrange Eq. (31) as follows:

$$z_2\left(t_{1Z}\right)^2+z_1\left(t_{1Z}\right)+z_0=0 \tag{32}$$

Applying the square roots solution, t_{1Z}* is found.

$$t_{1Z}{}^*=\frac{-z_1\pm\sqrt{z_1^2-4z_2z_0}}{2z_2} \tag{33}$$

Since $F(t_{1Z}) = (1 - e^{-\beta t1Z})$ falls within the interval of [0, 1], so does its complement $e^{-\beta t1Z}$. By rearranging Eq. (31), $e^{-\beta t1Z}$ becomes as follows:

$$e^{-\beta t_{1Z}}=\frac{\left[v_1\delta_1\left(1+\alpha_1\right)P_1\left(t_{1Z}\right)^2-\left(\delta_2+W_1\right)\delta_1\left(1+\alpha_1\right)P_1+2\lambda g v_1\left(t_{1Z}\right)+\left(\delta_3+G_3\right)\lambda g\right]}{\left\{\begin{array}{l}-\left(hg+G_3\right)\left[\beta\delta_1\left(1+\alpha_1\right)P_1\right]\left(t_{1Z}\right)^2+2v_1\lambda g\left(t_{1Z}\right)+v_1\lambda g\beta\left(t_{1Z}\right)^2\\+\left(\delta_3-hg\right)\left(\lambda g\beta\right)\left(t_{1Z}\right)-W_1\beta\delta_1\left(1+\alpha_1\right)P_1\left(t_{1Z}\right)+\left(\delta_2+W_1\right)\lambda\beta g\\+\left(\delta_3+G_3\right)\lambda g-W_1\left[\delta_1\left(1+\alpha_1\right)P_1+\beta\lambda g\right]-\left(hg+G_3\right)\lambda g\left(1-e^{-\beta t_{1Z}}\right)\end{array}\right\}} \tag{34}$$

*3.1. Recursive algorithm for finding t_{1Z}**

We propose the following recursive algorithm to locate t_{1Z}*:

(i) Let $e^{-\beta t1Z} = 0$ and $e^{-\beta t1Z} = 1$ initially, by applying Eq. (33) we obtain the upper and lower bounds of t_{1Z}* (i.e., t_{1ZU} and t_{1ZL}).

(ii) Applying the current values of t_{1ZU} and t_{1ZL} to update $e^{-\beta t1ZU}$ and $e^{-\beta t1ZL}$.
(iii) Re-apply Eq. (33) using the current $e^{-\beta t1ZU}$ and $e^{-\beta t1ZL}$ to recalculate the values for t_{1ZU} and t_{1ZL}.
(iv) Test for $t_{1ZU} = t_{1ZL}$ to see if it holds. If yes, the optimal t_{1Z}* is found. That is $t_{1Z}* = t_{1ZL} = t_{1ZU}$; otherwise, repeat on to step (ii).

4. Demonstration of the research result

To demonstrate the obtained research result's applicability and capability, this study supplies the following illustration with the assumption of related parameters exhibited in Table 1:

Table 1
Assumption of related parameters in this section

P_1	α_1	α_3	β_2	π	C	h	β	C_1	K_1	C_T
10000	0.5	0.1	0.5	0.4	\$2	\$0.4	1	\$2	\$90	\$0.01
K	α_2	x	β_1	g	C_S	λ	M	h_3	n	h_2
\$200	0.1	20%	-0.70	0.018	\$0.1	4000	\$2500	\$0.4	3	\$1.6

We start with verifying the convexity of $E[TCU(t_{1Z})]$, that is, to test whether $y(t_{1Z}) > t_{1Z} > 0$ (Eq. (A-8) in Appendix A) holds. Since $e^{-\beta t1Z}$ falls within [0, 1], let $e^{-\beta t1Z} = 0$ and $e^{-\beta t1Z} = 1$ initially, and apply Eq. (33) to arrive at $t_{1ZU} = 0.2113$ and $t_{1ZL} = 0.0731$. Next, compute $e^{-\beta t1ZU}$ and $e^{-\beta t1ZL}$ with t_{1ZU} and t_{1ZL}. Then, apply Eq. (A-8) using the current values of $e^{-\beta t1ZL}$, $e^{-\beta t1ZU}$, t_{1ZL}, and t_{1ZU} to confirm that $y(t_{1ZL}) = 0.2966 > t_{1ZL} = 0.0731 > 0$ and $y(t_{1ZU}) = 0.4513 > t_{1ZU} = 0.2113 > 0$, respectively. That is, in our example, for $\beta = 1$, $E[TCU(t_{1Z})]$ is convex. Therefore, the optimal t_{1Z}* exists. Table B-1 (in Appendix B) exhibits the additional convexity testing results with a broader range of β's, demonstrating our research result's more general applicability.

To seek for the optimal fabrication runtime t_{1Z}*, we apply the recursive algorithm provided in subsection 3.1. Table B-2 (in Appendix B) displays the detailed step-by-step outcomes from the proposed algorithm. It includes the starting values of upper and lower bounds of t_{1Z}, the optimal runtime $t_{1Z}* = 0.0905$ years, and $E[TCU(t_{1Z}*)] = \$13,268.47$.

4.1. Collective impact of our problem's key system features

Taking advantage of our model's capability, we explore the following collective impact of our problem's crucial system features. Figure 7 illustrates the combined impact of variations in the fabrication runtime t_{1Z} and the overtime added portion α_1 of output rate on $E[TCU(t_{1Z})]$. It exposes $E[TCU(t_{1Z})]$ significantly surges as α_1 rises, and $E[TCU(t_{1Z})]$ considerably increases as t_{1Z} deviates from the optimal point t_{1Z}* (i.e., 0.0905).

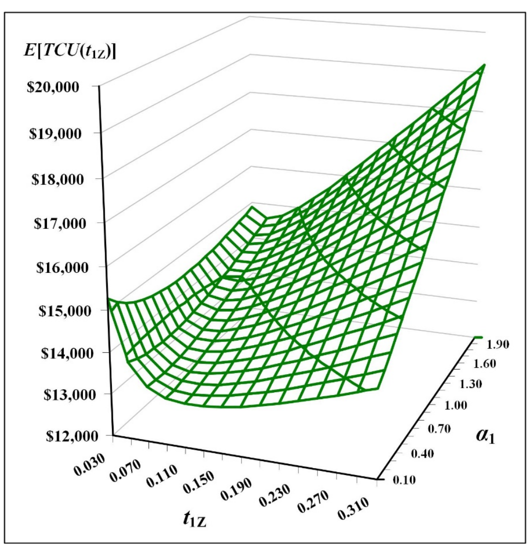

Fig. 7. The combined impact of variations in t_{1Z} and α_1 on $E[TCU(t_{1Z})]$

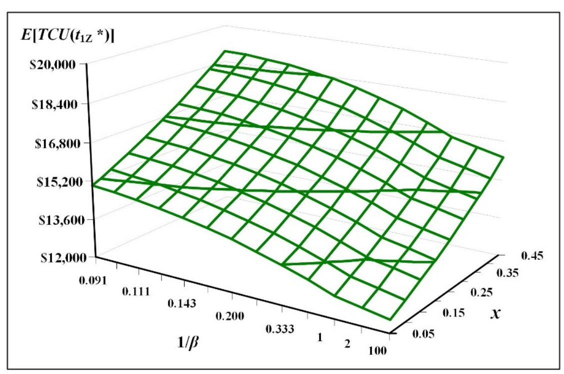

Fig. 8. Collective impact of variations in $1/\beta$ and x on $E[TCU(t_{1Z}*)]$

Fig. 8 displays the collective impact of variations in the mean-time-to-failure $1/\beta$ and x on $E[TCU(t_{1Z}*)]$. It indicates $E[TCU(t_{1Z}*)]$ harshly drops as $1/\beta$ rises to 0.17 and beyond, and $E[TCU(t_{1Z}*)]$ remarkably surges as x increases.

The joint impact of changes in π and x on t_{1Z}* are demonstrated in Figure 9. It exposes t_{1Z}* significantly declines as π increases; and when $\pi < 0.4$, t_{1Z}* knowingly increases as x rises; when $\pi \geq 0.4$, t_{1Z}* slightly increases as x goes up.

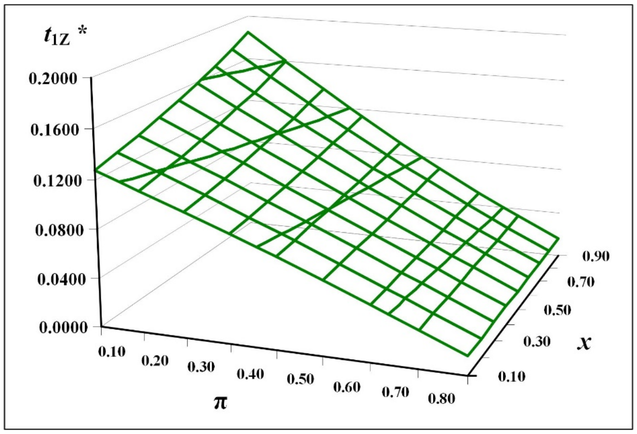

Fig. 9. The joint impact of changes in π and x on t_{1Z}*

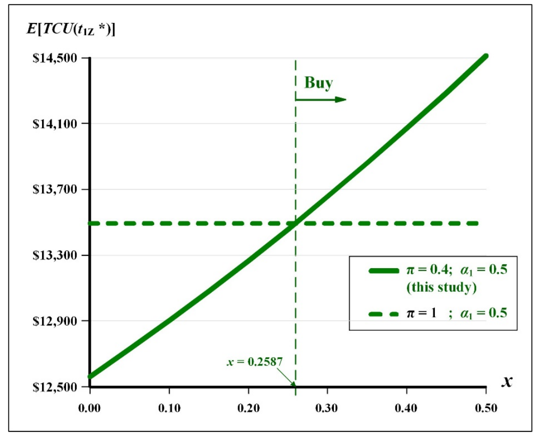

Fig. 10. Our study's critical x value

An in-depth explorative outcome discloses that our example's critical x value is 0.2587, as displayed in Figure 10. That means, in our example (with α_1 = 0.5), if the random scrap rate surges to 0.2587 and beyond, a more beneficial inventory replenishment decision is to choose a pure 'buy' strategy. Fig. 11 illustrates the collective impact of variations in n and π on $E[TCU(t_{1Z}*)]$. It exposes that $E[TCU(t_{1Z}*)]$ exceedingly surges as π goes up, and $E[TCU(t_{1Z}*)]$ significantly increases as n rises and deviates from 2.

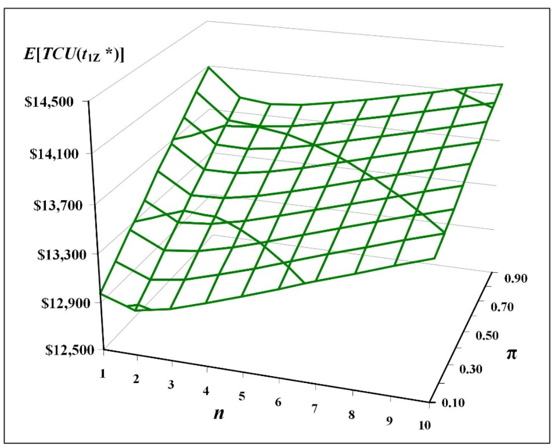

Fig. 11. The collective effect of changes in n and π on $E[TCU(t_{1Z}*)]$

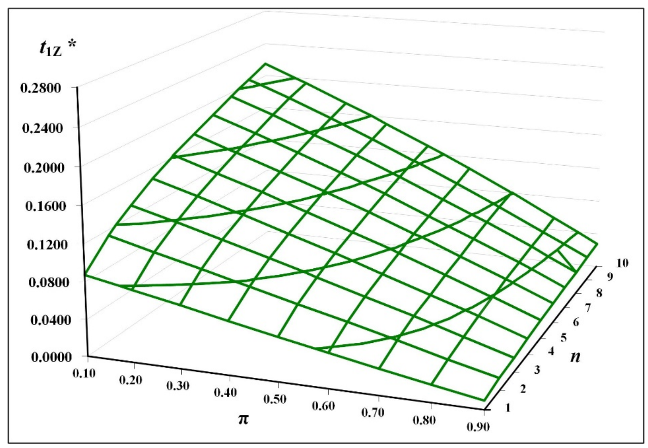

Fig. 12. Combined impact of changes in π and n on t_{1Z}*

Fig. 12 depicts the combined effect of changes in π and n on t_{1Z}*. It reveals t_{1Z}* drops remarkably as π rises, and t_{1Z}* greatly surges as n increases (especially when π ≤ 0.5). Fig. 13 exhibits the joint impact of changes in the overtime added portion α_1 of output rate and outsourcing proportion π on $E[TCU(t_{1Z}*)]$. It discloses that $E[TCU(t_{1Z}*)]$ remarkably surges as π goes up (especially when $\alpha_1 \leq 1.0$), and $E[TCU(t_{1Z}*)]$ considerably increases as α_1 rises (particularly when π ≤ 0.4).

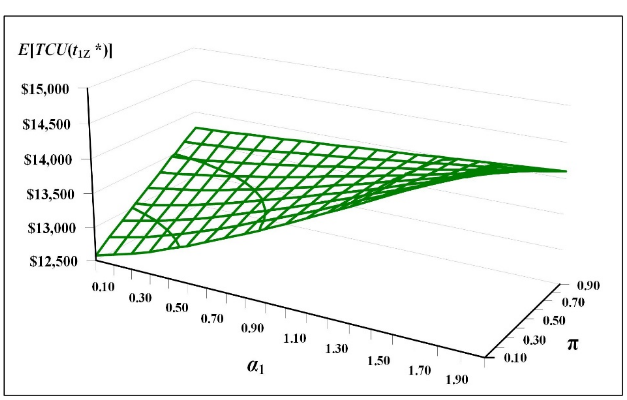

Fig. 13. Joint effect of variations in α_1 and π on $E[TCU(t_{1Z}*)]$

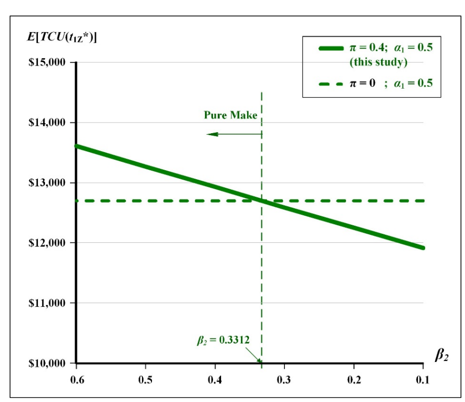

Fig. 14. The critical value of β_2 in our example

Furthermore, an in-depth explorative result reveals that our example's critical β_2 value is 0.3312, as displayed in Fig. 14. That means, in our example (with α_1 = 0.5), if the unit outsourcing add-up percentage increases to 33.12% and beyond, a more beneficial decision for replenishing stock is to choose a pure 'make ' strategy. Fig. 15 displays the collective influences of variations in π and overtime added portion α_1 of output rate on machine utilization. It shows the utilization exceedingly drops as π surges (particularly when $\alpha_1 \leq 1.0$). The utilization significantly reduces as α_1 rises (particularly for π ≤ 0.4).

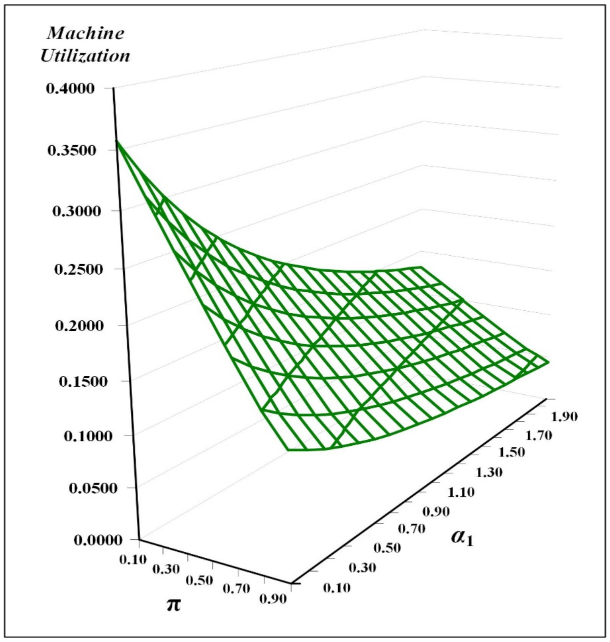

Fig. 15. The collective influences of variations in π and α_1 on machine utilization

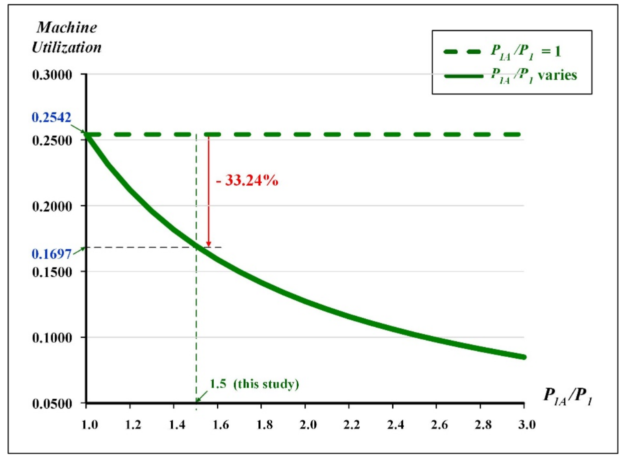

Fig. 16. Impact of changes in the ratio of (P_{1A} / P_1) on machine utilization

4.2. The impact of our problem's individual system feature

The impact of changes in the overtime output ratio (P_{1A} / P_1) on utilization is shown in Fig. 16. It reveals the utilization remarkably declines as (P_{1A} / P_1) rises. For (P_{1A} / P_1) = 1.5, it specifies the utilization drops from 0.2542 to 0.1697, a 33.24% decline due to our overtime strategy. Fig. 17 depicts the influence from π on utilization. It exhibits the utilization considerably reduces as π rises. For π = 0.4, it specifies that utilization declines a 42.44% reduction due to outsourcing implementation.

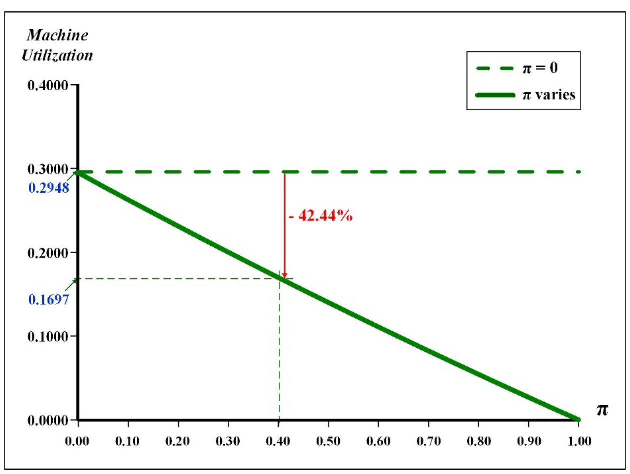

Fig. 17. The influence of changes in π on machine utilization

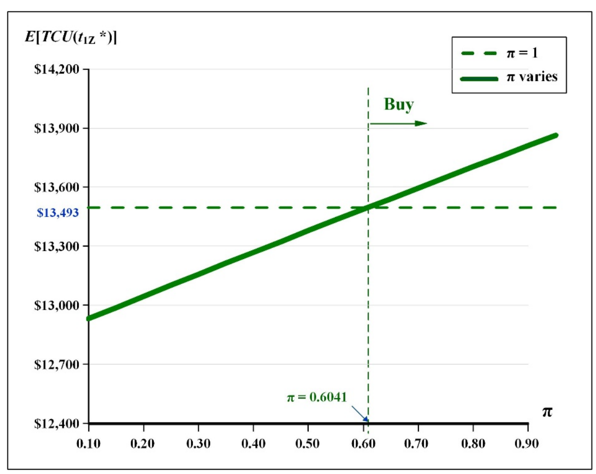

Fig. 18. A further investigative outcome relating to the critical π value

A further investigative outcome relating to the critical π value is exhibited in Fig. 18. It reveals the critical value of π is 0.6041. That is, when π increases to 0.6041 and beyond, a more beneficial inventory replenishing strategy is the pure 'buy.' A comparison of our proposed problem's utilization with other existing studies' results is conducted and illustrated in Fig. 19. Due to dual utilization-reduction policies (i.e., both outsourcing and overtime options), our utilization reduces to 0.1697 or 33.24% lower than that in a similar study with only outsourcing strategy (Chiu et al., 2020). Moreover, our model's utilization is 42.44% lower than that in a similar model with overtime strategy only. It is 61.54% less than a similar study without implementing neither outsourcing nor adjustable-rate strategies (Chiu et al., 2015). The prices we pay for the reducing utilization are 2.05%, 4.55%, and 8.41% increase in $E[TCU(t_{1Z}*)]$, respectively.

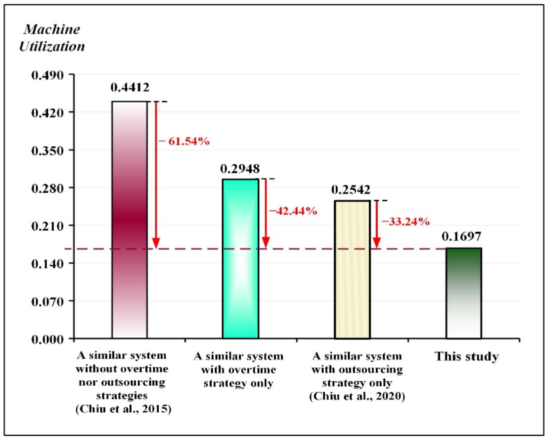

Fig. 19. A comparison of this study's utilization with other existing studies' results

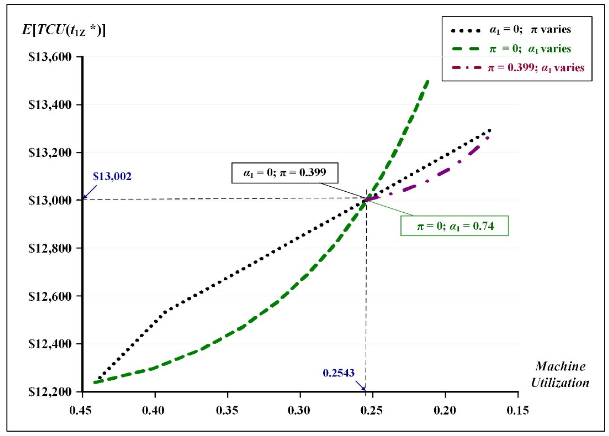

Fig. 20. Decision-support information relating to reducing utilization efficiently

Furthermore, we can also provide information to support managerial decision-making relating to efficient utilization reduction, as demonstrated in Figure 20. It discloses that the most beneficial way to decrease utilization starts with step (1) Applying the overtime strategy initially and increasing α_1 to 0.74, the utilization reduces to 0.2543, and $E[TCU(t_{1Z}*)]$ rises to $13,002 (see the green dash-line in Figure 20); (2) To further reduce machine utilization, the analytical result suggests us to switch to the combined strategies. Starting with $\pi = 0.399$ and $\alpha_1 = 0$; by keeping π at 0.399 and increasing α_1 (see the purple dash-line in Fig. 20). The aforementioned steps provide the most cost-effective way to reduce machine utilization.

5. Conclusions

This work builds an operating cost minimization model for buyer-vendor coordination batch system with scrap, breakdowns, overtime, multi-shipment, and an external source to assist the management in optimizing their production plan. The model simultaneously considers the expenses incurred in in-house manufacturing and inventory, finished goods' shipping, and stock holding at the retailer end. It also aims to achieve the operational goals of desirable quality, minimal production disruption, and shortening fabrication time to help minimize overall in-house operating costs and maximize customer satisfaction. Then, the model construction, cost analyses, differential calculus, and proposal of an algorithm help us derive the operational expense function, verify its convexity, and determine the best manufacturing runtime (refer to Sections 2 to 4). Lastly, to demonstrate how our study works, we provide a numerical example to illustrate our work's capability and applicability, as follows:

(1) Table B-1 (in Appendix B) exhibits that we can apply our work to a wide-ranging mean annual failure rate βs;
(2) Table B-2 exhibits the in-depth iterations for locating the optimal manufacturing runtime & Figure 7 shows our system cost's convexity;
(3) Fig. 8 to Fig. 15 depict the collective impact of key system factors (including $1/\beta$, π, n, α_1, β_2, and x) on the optimal operating expenditures, runtime decision, utilization, and make or buy decision making;
(4) Fig. 16 to Fig. 18 demonstrate the effects of individual system factors on the utilization and make-or-buy decision making;
(5) Fig. 19 compares utilization of our work with other existing prior results;
(6) Fig. 20 exposes the decision-support insight information concerning how to efficiently and economically reduce machine utilization.

Combining uncertain demand in the studied problem and exploring its impact on the results should be worth investigating for future work.

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Appendix – A

Deriving $E[TCU(t_{1Z})]$ (Eq. (30)) and its convexity.

Apply the E[x] to deal with the randomness, then, substitute Eq. (16), Eq. (21), and Eq. (27) in Eq. (26), with additional derivations $E[TCU(t_{1Z})]$ becomes as follows:

$$
\begin{aligned}
E\left[TCU\left(t_{1Z}\right)\right] &= \frac{\left\{\int_0^{t_{1Z}} E\left[TC\left(t_{1Z}\right)_1\right]\cdot f(t)\,dt + \int_{t_{1Z}}^{\infty} E\left[TC\left(t_{1Z}\right)_2\right]\cdot f(t)\,dt\right\}}{E[\boldsymbol{T}_Z]} \\
&= \left[\frac{\lambda}{\delta_1 + \frac{\lambda g\left(1-e^{-\beta t_{1Z}}\right)}{(t_{1Z})(1+\alpha_1)P_1}}\right]\left[\frac{\delta_2}{t_{1Z}} + \delta_3 + (t_{1Z})v_1\right] \\
&\quad + \left[\frac{\lambda}{\delta_1 + \frac{\lambda g\left(1-e^{-\beta t_{1Z}}\right)}{(t_{1Z})(1+\alpha_1)P_1}}\right]\left[\begin{array}{l} +\left[M + C_T(\lambda g) + C_1(\lambda g) + h_2\left(\frac{\lambda g^2}{2}\right) + h_3\left(\lambda g^2\right)\right]\left[\frac{1-e^{-\beta t_{1Z}}}{(t_{1Z})(1+\alpha_1)P_1}\right] \\ +h\cdot\frac{g}{(t_{1Z})}\left[\left(-e^{-\beta t_{1Z}}\right)(t_{1Z}) - \frac{1}{\beta}e^{-\beta t_{1Z}} + \frac{1}{\beta}\right] + (h_2 - h)\left(\frac{g}{2n}\right)v_2\left(1-e^{-\beta t_{1Z}}\right) \\ +\left(h_2 + 2h_3\right)\left(\frac{g}{2}\right)v_3\left(1-e^{-\beta t_{1Z}}\right) + h\cdot\left(\frac{g}{2}\right)v_2\left(1-e^{-\beta t_{1Z}}\right) \end{array}\right]
\end{aligned} \tag{A-1}
$$

where δ_1, δ_2, δ_3, v_1, v_2, and v_3 denote the following:

$$
\begin{aligned}
\delta_1 &= \left[\frac{1}{(1-\pi)} - E[x]\right] \\
\delta_2 &= \frac{(1+\beta_1)K}{(1+\alpha_1)P_1} + \frac{(1+\alpha_2)K}{(1+\alpha_1)P_1} + \frac{nK_1}{(1+\alpha_1)P_1} \\
\delta_3 &= (1+\beta_2)C\left(\frac{\pi}{1-\pi}\right) + (1+\alpha_3)C + C_T\left(\frac{1}{1-\pi}\right)\left[1 - E[x](1-\pi)\right] + C_S E[x]
\end{aligned} \tag{A-2}
$$

$$
\begin{aligned}
v_1 &= \frac{(1+\alpha_1)P_1}{(1-\pi)^2}\cdot\left[\begin{array}{l} \frac{h}{2\lambda}\left\{\left[1-E[x](1-\pi)\right]^2 + \frac{\lambda(1-\pi)\left[-\pi + E[x](1-\pi)\right]}{(1+\alpha_1)P_1}\right\} \\ +(h_2 - h)\left(\frac{1}{2\lambda n}\right)\left[1-E[x](1-\pi)\right]\left\{\left[1-E[x](1-\pi)\right] - \frac{\lambda(1-\pi)}{(1+\alpha_1)P_1}\right\} \\ +h_2\left(\frac{1}{2}\right)\left[1-E[x](1-\pi)\right]\left[\frac{1-\pi}{(1+\alpha_1)P_1}\right] \end{array}\right] \\
v_2 &= \left(\frac{1}{1-\pi}\right)\left\{\left[1-E[x](1-\pi)\right] - \frac{\lambda(1-\pi)}{(1+\alpha_1)P_1}\right\} \\
v_3 &= \left(\frac{1}{1-\pi}\right)\left\{\left[1-E[x](1-\pi)\right] + \frac{\lambda(1-\pi)}{(1+\alpha_1)P_1}\right\}
\end{aligned} \tag{A-3}
$$

Furthermore, suppose we let W_1, G_0, G_1, G_2 and G_3 represent the following:

$$\begin{aligned}
&W_1=\left[M+C_T(\lambda g)+C_1(\lambda g)+h_2\left(\frac{\lambda g^2}{2}\right)+h_3(\lambda g^2)\right]\frac{1}{(1+\alpha_1)P_1}+\frac{h\cdot g}{\beta}\\
&G_0=(h_2-h)\left(\frac{g}{2n}\right)v_2;\ G_1=(h_2+2h_3)\left(\frac{g}{2}\right)v_3;\ G_2=h\cdot\left(\frac{g}{2}\right)v_2\\
&G_3=G_0+G_1+G_2
\end{aligned}\tag{A-4}$$

Eq. (A-1) (i.e., $E[TCU(t_{1Z})]$) becomes (as expressed in Eq. (30)):

$$E\left[TCU(t_{1Z})\right]=\left[\frac{\lambda}{\delta_1+\dfrac{\lambda g\left(1-e^{-\beta t_{1Z}}\right)}{(t_{1Z})(1+\alpha_1)P_1}}\right]\begin{bmatrix}\dfrac{\delta_2}{t_{1Z}}+\delta_3+v_1(t_{1Z})+\dfrac{W_1}{t_{1Z}}-hg\left(e^{-\beta t_{1Z}}\right)-\dfrac{W_1}{(t_{1Z})}\left(e^{-\beta t_{1Z}}\right)\\ +G_3\left(1-e^{-\beta t_{1Z}}\right)\end{bmatrix}\tag{A-5}$$

Apply the first- and second-derivative of $E[TCU(t_{1Z})]$, we obtain equations (A-6) and (A-7) below:

$$\begin{aligned}
\frac{dE\left[TCU(t_{1Z})\right]}{d(t_{1Z})}&=\frac{\lambda(1+\alpha_1)P_1}{\left(\delta_1(t_{1Z})(1+\alpha_1)P_1+\lambda g\left(1-e^{-\beta t_{1Z}}\right)\right)^2}\cdot\\
&\begin{bmatrix}
-(\delta_2+W_1)\left(\delta_1(1+\alpha_1)P_1+\lambda\beta e^{-\beta t_{1Z}}g\right)\\
-(\delta_3+G_3)\cdot\lambda g\left(\beta e^{-\beta t_{1Z}}t_{1Z}+e^{-\beta t_{1Z}}-1\right)\\
-(hg+G_3)\left(-\beta\delta_1 e^{-\beta t_{1Z}}(t_{1Z})^2(1+\alpha_1)P_1-\beta\lambda g e^{-\beta t_{1Z}}(t_{1Z})-\lambda g e^{-2\beta t_{1Z}}+\lambda g e^{-\beta t_{1Z}}\right)\\
+v_1\left(\delta_1(t_{1Z})^2(1+\alpha_1)P_1-\lambda g\beta e^{-\beta t_{1Z}}(t_{1Z})^2-2\lambda g e^{-\beta t_{1Z}}(t_{1Z})+2\lambda g(t_{1Z})\right)\\
+e^{-\beta t_{1Z}}W_1\left(\delta_1(1+\alpha_1)P_1+\beta\delta_1(t_{1Z})(1+\alpha_1)P_1+\beta\lambda g\right)
\end{bmatrix}
\end{aligned}\tag{A-6}$$

$$\begin{aligned}
\frac{d^2E\left[TCU(t_{1Z})\right]}{d(t_{1Z})^2}&=\lambda\left(\frac{(1+\alpha_1)P_1}{\left(\delta_1(t_{1Z})(1+\alpha_1)P_1+\lambda g\left(1-e^{-\beta t_{1Z}}\right)\right)^3}\right)\cdot\\
&\begin{bmatrix}
(\delta_2+W_1)\begin{pmatrix}\lambda^2g^2\beta^2e^{-\beta t_{1Z}}+\lambda^2g^2\beta^2e^{-2\beta t_{1Z}}+4\delta_1\lambda g\beta e^{-\beta t_{1Z}}(1+\alpha_1)P_1\\ +2\delta_1^2\left[(1+\alpha_1)P_1\right]^2+\delta_1\lambda g\beta^2e^{-\beta t_{1Z}}(1+\alpha_1)P_1(t_{1Z})\end{pmatrix}\\
+\lambda g(\delta_3+G_3)\begin{pmatrix}\lambda g\beta^2e^{-2\beta t_{1Z}}(t_{1Z})+2\lambda g\beta e^{-2\beta t_{1Z}}-2\delta_1(1+\alpha_1)P_1+\lambda g\beta^2e^{-\beta t_{1Z}}(t_{1Z})-2\lambda g\beta e^{-\beta t_{1Z}}\\ +2\beta\delta_1e^{-\beta t_{1Z}}(1+\alpha_1)P_1(t_{1Z})+\beta^2\delta_1e^{-\beta t_{1Z}}(1+\alpha_1)P_1(t_{1Z})^2+2\delta_1e^{-\beta t_{1Z}}(1+\alpha_1)P_1\end{pmatrix}\\
+e^{-2\beta t_{1Z}}(-hg-G_3)\begin{pmatrix}\beta^2\lambda^2g^2(t_{1Z})+2\beta\lambda^2g^2-2\beta\lambda^2g^2e^{\beta t_{1Z}}+\beta^2\lambda^2g^2e^{\beta t_{1Z}}(t_{1Z})\\ +\beta^2\delta_1\lambda g(t_{1Z})^2(1+\alpha_1)P_1+4\beta\delta_1\lambda g(t_{1Z})(1+\alpha_1)P_1+2\delta_1\lambda g(1+\alpha_1)P_1\\ -2\beta\delta_1\lambda g e^{\beta t_{1Z}}(t_{1Z})(1+\alpha_1)P_1+2\beta^2\delta_1\lambda g e^{\beta t_{1Z}}(t_{1Z})^2(1+\alpha_1)P_1\\ +\beta^2\delta_1^2e^{\beta t_{1Z}}(t_{1Z})^3\left[(1+\alpha_1)P_1\right]^2-2\delta_1\lambda g e^{\beta t_{1Z}}(1+\alpha_1)P_1\end{pmatrix}\\
+\lambda g v_1\begin{pmatrix}2\lambda g e^{-2\beta t_{1Z}}+2\lambda g+\lambda g\beta^2e^{-\beta t_{1Z}}(t_{1Z})^2+\lambda g\beta^2e^{-2\beta t_{1Z}}(t_{1Z})^2+4\lambda g\beta e^{-2\beta t_{1Z}}(t_{1Z})\\ -4\lambda g\beta e^{-\beta t_{1Z}}(t_{1Z})-4\lambda g e^{-\beta t_{1Z}}+\delta_1\beta^2e^{-\beta t_{1Z}}(t_{1Z})^3(1+\alpha_1)P_1\end{pmatrix}\\
-e^{-\beta t_{1Z}}W_1\begin{pmatrix}2\delta_1^2\left[(1+\alpha_1)P_1\right]^2+\beta^2\delta_1^2(t_{1Z})^2\left[(1+\alpha_1)P_1\right]^2+2\beta\delta_1^2(t_{1Z})\left[(1+\alpha_1)P_1\right]^2\\ +\beta^2\delta_1\lambda g e^{-\beta t_{1Z}}(t_{1Z})(1+\alpha_1)P_1+2\beta^2\delta_1\lambda g(t_{1Z})(1+\alpha_1)P_1+2\beta\delta_1\lambda g(1+\alpha_1)P_1\\ +\beta^2\lambda^2g^2+2\beta\delta_1\lambda g e^{-\beta t_{1Z}}(1+\alpha_1)P_1+\beta^2\lambda^2g^2e^{-\beta t_{1Z}}\end{pmatrix}
\end{bmatrix}
\end{aligned}\tag{A-7}$$

If $y(t_{1Z}) > t_{1Z} > 0$ holds, then $E[TCU(t_{1Z})]$ is convex (because the first term on RHS of Eq. (A-7) is positive, so if the second term is also positive, then $E[TCU(t_{1Z})]$ is convex).

$$y\left(t_{1\mathrm{z}}\right)=\frac{\left[\begin{array}{l}\left(\delta_2+W_1\right)\left(\lambda^2 g^2\beta^2 e^{-2\beta t_{1z}}+\lambda^2 g^2\beta^2 e^{-\beta t_{1z}}+4v_0\lambda g\beta e^{-\beta t_{1z}}\left(1+\alpha_1\right)P_1+2{v_0}^2\left[\left(1+\alpha_1\right)P_1\right]^2\right)\\ +\lambda g\left(2\lambda g\beta e^{-2\beta t_{1z}}-2\lambda g\beta e^{-\beta t_{1z}}+2\delta_1 e^{-\beta t_{1z}}\left(1+\alpha_1\right)P_1-2\delta_1\left(1+\alpha_1\right)P_1\right)\left(\delta_3+G_3\right)\\ +e^{-2\beta t_{1z}}\left(-hg-G_3\right)\left(-2\beta\lambda^2 g^2 e^{\beta t_{1z}}2\beta\lambda^2 g^2-2\delta_1\lambda g e^{\beta t_{1z}}\left(1+\alpha_1\right)P_1+2\delta_1\lambda g\left(1+\alpha_1\right)P_1\right)\\ +\lambda g v_1\left(2\lambda g-4\lambda g e^{-\beta t_{1z}}+2\lambda g e^{-2\beta t_{1z}}\right)\\ -e^{-\beta t_{1z}}W_1\left(2\beta\delta_1\lambda g\left(1+\alpha_1\right)P_1+2{\delta_1}^2\left[\left(1+\alpha_1\right)P_1\right]^2+\beta^2\lambda^2 g^2+2\beta\delta_1\lambda g e^{-\beta t_{1z}}\left(1+\alpha_1\right)P_1+\beta^2\lambda^2 g^2 e^{-\beta t_{1z}}\right)\end{array}\right]}{-\left[\begin{array}{l}\left(\delta_1\lambda g\beta^2 e^{-\beta t_{1z}}\left(1+\alpha_1\right)P_1\right)\left(\delta_2+W_1\right)\\ +\lambda g\left(\delta_3+G_3\right)\left(\lambda g\beta^2 e^{-2\beta t_{1z}}+\beta^2\delta_1 e^{-\beta t_{1z}}\left(1+\alpha_1\right)P_1\left(t_{1Z}\right)+\lambda g\beta^2 e^{-\beta t_{1z}}+2\beta\delta_1 e^{-\beta t_{1z}}\left(1+\alpha_1\right)P_1\right)\\ +e^{-2\beta t_{1z}}\left(-hg-G_3\right)\left(\begin{array}{l}\beta^2\lambda^2 g^2+\beta^2\delta_1\lambda g\left(t_{1Z}\right)\left(1+\alpha_1\right)P_1+\beta^2\lambda^2 g^2 e^{\beta t_{1z}}+2\beta^2\delta_1\lambda g e^{\beta t_{1z}}\left(t_{1Z}\right)\left(1+\alpha_1\right)P_1\\ -2\beta\delta_1\lambda g e^{\beta t_{1z}}\left(1+\alpha_1\right)P_1+4\beta\delta_1\lambda g\left(1+\alpha_1\right)P_1+\beta^2{\delta_1}^2 e^{\beta t_{1z}}\left(t_{1Z}\right)^2\left[\left(1+\alpha_1\right)P_1\right]^2\end{array}\right)\\ +\lambda g v_1\left(\lambda g\beta^2 e^{-2\beta t_{1z}}\left(t_{1Z}\right)+\lambda g\beta^2 e^{-\beta t_{1z}}\left(t_{1Z}\right)+4\lambda g\beta e^{-2\beta t_{1z}}-4\lambda g\beta e^{-\beta t_{1z}}+\delta_1\beta^2 e^{-\beta t_{1z}}\left(t_{1Z}\right)^2\left(1+\alpha_1\right)P_1\right)\\ -e^{-\beta t_{1z}}W_1\left(\beta^2{\delta_1}^2\left(t_{1Z}\right)\left[\left(1+\alpha_1\right)P_1\right]^2+2\beta{\delta_1}^2\left[\left(1+\alpha_1\right)P_1\right]^2+2\beta^2\delta_1\lambda g\left(1+\alpha_1\right)P_1+\beta^2\delta_1\lambda g e^{-\beta t_{1z}}\left(1+\alpha_1\right)P_1\right)\end{array}\right]}>t_{1\mathrm{z}}>0 \qquad \text{(A-8)}$$

Appendix – B

Table B-1

Additional convexity testing results with a broader range of β*'s*

β	$\gamma(t_{1ZU})$	t_{1ZU}	$\gamma(t_{1ZL})$	t_{1ZL}
12	0.4961	0.2079	0.0402	0.0184
9	0.3883	0.2080	0.0523	0.0236
6	0.3283	0.2082	0.0747	0.0327
3	0.3226	0.2088	0.1311	0.0506
2	0.3505	0.2095	0.1778	0.0603
1	**0.4513**	**0.2113**	**0.2966**	**0.0731**
0.5	0.6501	0.2150	0.5020	0.0808
0.01	4.6081	0.4487	4.2414	0.0892

Table B-2

Detailed step-by-step outcomes from the proposed algorithm for locating $t_{1Z}*$

Step	t_{1ZU}	$e^{-\beta t1ZU}$	t_{1ZL}	$e^{-\beta t1ZL}$	t_{1ZU} - t_{1ZL}	$E[TCU(t_{1ZU})]$	$E[TCU(t_{1ZL})]$
-	**-**	0	**-**	1	**-**	**-**	**-**
1	**0.2113**	0.8095	**0.0731**	0.9295	0.1382	$14,057.20	$13,315.62
2	0.1087	0.8970	0.0874	0.9163	0.0213	$13,303.39	$13,269.68
3	0.0935	0.9107	0.0899	0.9140	0.0036	$13,269.61	$13,268.51
4	0.0910	0.9130	0.0904	0.9136	0.0006	$13,268.51	$13,268.47
5	0.0906	0.9134	0.0904	0.9135	0.0002	$13,268.47	$13,268.47
6	**0.0905**	0.9135	**0.0905**	0.9135	**0.0000**	**$13,268.47**	**$13,268.47**