International Journal of Industrial Engineering Computations 14 (2023) 523–538

Contents lists available at GrowingScience

International Journal of Industrial Engineering Computations

homepage: www.GrowingScience.com/ijiec

A joint replenishment problem with the (T, k_i) policy under obsolescence

Ricardo Afonso[a,b]*, Pedro Godinho[a] and João Paulo Costa[a]

[a]Univ Coimbra, CeBER, Faculty of Economics, Av Dias da Silva 165, 3004-512 Coimbra, Portugal
[b]Altice Labs, S.A., Rua Eng. Jose Ferreira Pinto Basto, 3810-106, Aveiro, Portugal

C H R O N I C L E

Article history:
Received October 2 2022
Received in Revised Format March 5 2023
Accepted April 7 2023
Available online April, 7 2023

Keywords:
Inventory lot sizing
Joint replenishment problem
JRP
Obsolescence

A B S T R A C T

Companies are frequently confronted with the need to order different types of items from a single supplier or to manufacture the items in a production line. Indeed, coordinated ordering of multiple items may lead to important savings whenever a family of items can be ordered from a common supplier, produced in a common facility, or use a common mode of transportation. The Joint Replenishment Problem (JRP) tackles the coordinated replenishment of multiple items by minimizing the total cost, composed of ordering (or setup) costs and holding costs, while satisfying the demand. On the other hand, when items are subject to obsolescence, they may face an abrupt decline in demand as they are no longer needed. This decline can be caused by reasons such as rapid advancements in technology, going out of fashion, or ceasing to be economically viable. The present article develops an extension of the JRP where the items may suddenly become obsolete during an infinite planning horizon. The point at which an item becomes obsolete is uncertain. The lifetimes of the items are assumed to follow independent negative exponential distributions. A model is proposed by using the total expected discounted cost as the minimization criterion. The time value of money is considered through an appropriate discount rate. Extensive tests were performed to assess the impact of obsolescence rates and discount rates on the ordering policies. The progressive increase of the obsolescence rates determines smaller periods between successive replenishments as well as smaller lot sizes. The same impact occurs when we face a progressive increase of the discount rate.

1. Introduction

Some of the first known inventory lot sizing models in the literature, such as the Economic Order Quantity (EOQ) model by Harris (1913) or the Economic Production Quantity (EPQ) model by Taft (1918, *apud* Holmbom & Segerstedt, 2014), were developed to find optimal ordering policies for a single item. In simple terms, these models rely on the minimization of the setup and holding costs by determining when and how much to order while satisfying the anticipated demand for the single item.

However, the assumption of one single item may be restrictive when companies are, for example, confronted with the need to order different items from a single supplier. Indeed, coordinated ordering may lead to a reduction on fixed costs, for instance, by filling a truckload or by substantially lowering the setup costs if a group of products are manufactured together in a production line (Axsäter, 2015). In these circumstances, the Joint Replenishment Problem (JRP), tackling the coordinated replenishment of different types of items (or products) in the same order, arises as a natural option to cope with inventory lot sizing problems involving multiple items. This problem usually applies to items that are provided by the same supplier, or to the determination of lot sizes and schedule of multiple items in single-facility production/inventory systems (Lee & Yao, 2003).

* Corresponding author Tel.: +351 962 021 220
E-mail: uc1995027815@student.uc.pt (R. Afonso)
ISSN 1923-2934 (Online) - ISSN 1923-2926 (Print)
2023 Growing Science Ltd.
doi: 10.5267/j.ijiec.2023.4.002

Another aspect that can impact the decisions with respect to inventory lot sizing problems is the possibility of the items becoming obsolete. An additional source of complexity is sometimes introduced in the system when the point at which an item will become obsolete cannot be predicted in advance. Items subject to obsolescence may experience an abrupt decline in demand because they are no longer needed due to, for example, an advance in technology, going out of fashion, or because the items cease to be economically viable.

Several studies can be found in the literature where items are subject to obsolescence, such as Masters (1991), Joglekar & Lee (1993) and van Delft & Vial (1996), among others. Goyal & Giri (2001) refer that very few problems on obsolescence have been addressed by the researchers. Moreover, most of the studies assuming obsolescence of the items consider just one type of item, instead of the more complex setting of problems involving multiple items. As far as multi-item inventory problems are concerned, there are studies pointing out the lack of investigation tackling the JRP where the items may be subject to obsolescence (Khouja & Goyal, 2008). To address this lack of investigation, Afonso et al. (2022) developed an approximate model of the JRP where items may become obsolete at some random future time, meaning that the lifetimes of the different items are uncertain.

The contribution of the present study is the development of a precise model of the JRP where items are prone to obsolescence. The model considers the following assumptions: constant demand, no quantity discounts, no shortages allowed, linear holding cost, zero lead time and instantaneous delivery. The lifetimes of the items or, in other words, the obsolescence time of the items, follow negative exponential distributions. The use of this distribution for modelling obsolescence is justified by Masters (1991) with the following argument: “although many distributions are plausible, the (negative) exponential (distribution) is appropriate for sudden obsolescence phenomenon since it models a constant obsolescence rate”, i.e., “the age of the item does not influence the probability of obsolescence during any subsequent interval”, which means that the obsolescence process has no memory and a time-invariant lot size is optimal. We also incorporate the time value of money in the proposed model by considering the discount rate.

The article is structured as follows. Section 2 presents a literature review about inventory lot sizing problems where items may become obsolete and addresses this stream under the JRP framework. In Section 3 we develop the total expected discounted JRP obsolescence cost model through a model minimizing the setup, acquisition and holding costs along the infinite planning horizon. The optimization process is performed through a recursive procedure. Section 4 presents and discusses numerical examples. Finally, in Section 5 some conclusions are summarized.

2. Obsolescence and JRP

Many inventory lot sizing problems in the literature assume that stocked items have infinite shelf lives. This may not be adequate when we are confronted with real life situations where items may deteriorate (e.g. alcohol, medicines) or become obsolete (e.g. technology, fashion) over time. Deterioration refers to the damage, spoilage, dryness, vaporization, etc. of the products (Goyal & Giri, 2001; Bakker et al., 2012). Obsolescence differs from deterioration because items subject to obsolescence are likely to lose their value over time or to face the problem of finite shelf lives due to advancements in technology, change of consumer tastes, introduction of new products (for products such as electronics and apparel), among others. According to Pahl & Voβ (2014), obsolescence has to do with the products whose functionality does not degrade, but where demand deteriorates over time as customers’ perceived utility decreases. Such loss of utility may be relatively swift, with demand dropping suddenly to zero (Arcelus et al., 2002). Obsolescence can be classified in two types (Sandborn, 2013): procurement obsolescence and sudden obsolescence. The type addressed herein is the sudden obsolescence, which refers to the obsolescence of an inventory of items that remain after the demand for the item disappears (Brown et al., 1964 *apud* Bartels et al., 2012). This may happen when the product design or system specifications change and, as a result, existing inventories (of products or their parts) are no longer needed, for example because the product is replaced by a substitute one performing similar or identical functions, or the like (Arcelus et al., 2002). Hence, a partial or total loss of value of the inventory on hand can be anticipated as a result of obsolescence (van Delft & Vial, 1996).

According to Nahmias (2011), obsolescence is typically characterized by uncertainty in the lifetime of a product as the point at which an item becomes obsolete cannot be predicted in advance. In this way, probability distributions are used to describe the lifetime of items subject to obsolescence. Indeed, different probability distributions can be found in the literature to model obsolescence (Dohi & Osaki, 1995; Covert & Philip, 1973; Wee,1997). The model proposed in this study assumes the negative exponential probability distribution to model the lifetime of items subject to obsolescence, similarly to Masters (1991), Joglekar & Lee (1993) and van Delft & Vial (1996). Masters (1991) argues that the negative exponential probability distribution is appropriate for modelling the lifetime of an item subject to sudden obsolescence. van Delft & Vial (1996) obtained the optimal ordering policy by discounting costs through the discount rate of the firm and by performing a probabilistic analysis based on whether obsolescence occurs during or after an inventory cycle. The analysis performed in Section 3 undertakes this kind of approach.

Despite obsolescence having been considered in inventory lot sizing problems, there are authors still mentioning some lack of research in this area (Goyal & Giri, 2001; Khanlarzade et al., 2014). Khouja & Goyal (2008) refer to the usefulness of exploring extensions of the JRP tackling product obsolescence. More recently, in their literature review on the JRP, Bastos et al. (2017) still do not refer to any study involving obsolescence. However, the impact of incorporating the risk of obsolescence in inventory decisions can determine significant savings for the companies (Song & Zipkin, 1996).

To the best of our knowledge, most of the inventory lot sizing studies assuming obsolescence consider one single-item. Few studies are found involving multiple items subject to either deterioration or (sudden) obsolescence. Afonso et al. (2022) identify some studies with respect to the JRP where items are subject to deterioration, but none in the context of sudden obsolescence.

The classical JRP can be interpreted as an extension of the classical EOQ involving multiple items where the ordering policy is determined through the optimal trade-off between ordering costs and holding costs (Afonso et al., 2022). The cost of placing an order to the supplier in the JRP context typically includes two components, one major ordering cost (or major setup cost) independent of the number of different items in the order, and several minor ordering costs (or minor setup costs) that depend on the number of different items included in the order (Khouja & Goyal, 2008).

Under the JRP, it makes sense to consider the (T, k_i) policy constituted by a cycle of T units of time and a set of integer multipliers k_i where the time interval between successive replenishments of the item i is given by $T_i = k_i T$. In other words, the cycle time of an item i is an integer multiple k_i of the cycle such that $T_i = k_i T$ and the corresponding order quantity is given by $Q_i = T_i D_i = k_i T D_i$.

The classical JRP does not consider the acquisition costs (Khouja & Goyal, 2008). This component of costs is typically ignored since it can be understood as a sunk cost due to the assumption that all demand must be fully satisfied (Berk & Gurler, 2017). However, one must note that this should not be the case when the time value of money is considered because the moment at which the acquisition occurs may have a relevant impact on the present value of the total costs. The model developed in Section 3 takes into account the time value of money and, consequently, does not ignore the acquisition costs.

3. The total discounted JRP obsolescence cost model

The total loss single-item obsolescence model described by van Delft and Vial (1996) was extended by Afonso et al. (2022) to deliver an approximate model of the JRP to multiple items subject to obsolescence. In this study, we follow a similar strategy to propose a precise model of the JRP with items subject to obsolescence. The total loss occurs at sudden obsolescence, meaning that all items on hand instantaneously lose their value at once.

3.1 Assumptions and notation

The lifetimes of the multiple items are assumed to be independent and identically distributed and to follow negative exponential distributions. The time value of money is considered through an appropriate discount rate. We also assume, in our model: instantaneous delivery (i.e., replenishment rate is infinite), no quantity discounts, shortages are not allowed, constant demand rate, constant unit cost, linear holding cost and fixed ordering cost, which is described as the sum of two components: a major and a minor setup cost. Without loss of generality, the lead time is equal to zero.

The model considers the (T, k_i) replenishment policy which consists of a common cycle of T units of time and a set of integer multipliers, one multiplier k_i for each item, corresponding to the number of cycles between replenishments of that item (Khouja & Goyal, 2008; Silver et al., 2017).

The notation of the model is:

N – number of items;

$i = 1, 2, \dots, N$ – the item index;

S – the ordered set of N items;

A – major ordering cost (major setup cost) associated with each replenishment;

a_i – minor ordering cost incurred if item i is ordered in a replenishment;

c_i – cost per unit of item i;

h_i – unit holding cost of item i per unit of time;

D_i – demand of item i per unit of time;

$H_i(t)$ – expected discounted holding cost for item i incurred during t units of time, where the stock level at the beginning satisfies the demand during that period;

B – a nonempty ordered subset of S (i.e., $B \subseteq S \text{ and } B \neq \emptyset$);

$\Gamma(B)$ – time between successive replenishments (or cycle length) expressed in units of time, involving only the items that do belong to B;

T – time between successive replenishments (or cycle length) expressed in units of time when $B = S$, i.e., $T = \Gamma(S)$;

Q_i – order quantity of item i (quantity of items of type i to order);

δ – discount rate;

θ_i – rate of obsolescence of item i;

$C_o(B)$ – setup costs incurred during the cycle, involving only the items that do belong to B;

$C_a(B)$ – acquisition costs incurred during the cycle, involving only the items that do belong to B;

$C_h(B)$ – expected discounted holding costs incurred during the cycle, involving only the items that do belong to B;

$\mathrm{Z}_0(B)$ – expected costs incurred during the cycle, involving only the items that do belong to B;

$\mathrm{Z}_m(B)$ – expected costs incurred after the cycle where m items that do belong to B do not become obsolete during the cycle or, in other words, where m items survive the cycle, with $1 \leq m \leq card(B) \leq N$;

$V(B)$ – total expected discounted infinite horizon cost associated to the items that do belong to B.

According to Afonso et al. (2022), $H_i(t)$ is given by Eq. (1).

$$H_i(t) = h_i \int_0^t e^{-\delta x}(D_i t - D_i x)\theta_i e^{-\theta_i x} dx = h_i \theta_i \left(\frac{D_i t}{\delta + \theta_i} + \frac{D_i \left(e^{-(\delta+\theta_i)t} - 1\right)}{(\delta + \theta_i)^2} \right) \tag{1}$$

Next, we present a model of the expected discounted cost in a JRP, when the items are prone to obsolescence. For the sake of clarity and convenience, the model is developed in two steps. 1) In the first step, the integer multipliers are simply ignored, which means that all the items are included in each and every order if they have not yet become obsolete. In other words, all the integer multipliers are implicitly assumed to be equal to one, i.e., $k_i = 1, \forall_i$. 2) In the second step, the model is extended in order to take on board the integer multipliers.

In the Subsection 0, we formulate the general expression of $V(B)$ without the integer multipliers. Subsection 3.2 extends the model developed in Subsection 0 in order to incorporate the integer multipliers. Subsection 3.3 presents a procedure to determine the optimal strategy according to the model.

3.1 Expression of the objective function without the integer multipliers

The ultimate objective is to determine the optimal value of the decision variable T by minimizing $V(S)$, i.e., when $B = S$. For convenience, the formulation is first derived for any subset B of S. The expression of $V(S)$ is obtained by simply replacing B with S. The derivation of $V(S)$ in this manner helps describing the procedure presented in Section 3.3.

Some aspects of the structure of the optimal policy of our obsolescence model can be specified *a priori*. In fact, one must note that under instantaneous delivery, a given item i is ordered only if its inventory level is zero. Assuming that the inventory level is zero at some time t, the corresponding optimal lot size is Q_i such that $T = \frac{Q_i}{D_i}, 1 \leq i \leq N$ (recall that ignoring the integer multipliers, all items are ordered in each period of T units of time). Then, if obsolescence does not occur during the period $\left[t, t + \frac{Q_i}{D_i}\right]$, and considering the memoryless property of the negative exponential distribution, used to model the obsolescence lifetime of the items, the inventory system, with respect to any item i, is in the same status at time $t + T$ as it was at time t, keeping Q_i as the optimal lot size.

Hence, in a similar way as developed by van Delft & Vial (1996) and Afonso et al. (2022), the computation of the cost components of the model objective function follows an approach that considers two alternatives: costs incurred when obsolescence occurs during the first cycle or after it. Let us assume B such that $card(B) = b$ and $B = (i_1, i_2, \ldots, i_b)$.

Costs incurred when obsolescence occurs during the first cycle

Considering the subset B, if obsolescence of all the items occurs during the first cycle, there are no other costs than the expected ones incurred at the beginning of the cycle:

$$\mathrm{Z}_0(B) = C_o(B) + C_a(B) + C_h(B) \tag{2}$$

Note that the setup and acquisition cost components $C_o(B)$ and $C_a(B)$ correspond to costs incurred at the beginning of the cycle, because, according to the assumptions, we need to have at this moment the necessary quantities of items to satisfy the demand, while $C_h(B)$ corresponds to the discounted expected holding costs incurred during the cycle. Thus, such as considered by Afonso et al. (2022), $C_o(B)$, $C_a(B)$ and $C_h(B)$ are incurred regardless the occurrence of obsolescence during the cycle, i.e., these costs occur with probability one. Since we are disregarding the integer multipliers, these three cost components are fully incurred as all the items are ordered in each replenishment. Then, the detailed expressions of $C_o(B)$ and $C_a(B)$, incurred at the beginning of the cycle, are given by $A + \sum_{i \in \mathrm{B}} a_i$ and $\sum_{i \in \mathrm{B}} c_i D_i \Gamma(B)$, respectively. The detailed expression of $C_h(B)$, incurred during the cycle of $\Gamma(B)$ units of time, but discounted at the beginning of this cycle, is given by $\sum_{i\in \mathrm{B}} H_i(\Gamma(B)) = \sum_{i\in \mathrm{B}} h_i\theta_i\left(\frac{D_i\Gamma(B)}{\delta+\theta_i}+\frac{D_i\left(e^{-(\delta+\theta_i)\Gamma(B)}-1\right)}{(\delta+\theta_i)^2}\right)$. Note that when $B = S$, the sums occur over the N items and $\Gamma(B) = T$.

Total costs

To determine the costs after the first cycle, i.e., after moment $t = \Gamma(B)$, it is necessary to consider all the possible combinations where at least one item survives the first cycle. Thus, since $card(B) = b$, this means that we need to consider the cases where just one of the items survives and the other $b - 1$ do not, the cases where just two any items survive and the other $b - 2$ do not, and so on, until the last case where all the items of B do survive the cycle of $\Gamma(B)$ units of time. To properly determine the costs of all these cases, each of them must consider the multiplication of two factors: 1) one factor involving the probabilities of survival of the items, and 2) another factor corresponding to the optimal expected discounted cost of the items that survived. Please note that for the items that survived the first cycle, the model recursively considers they will be included in another 'first' cycle, T units of time ahead, with the cost adjusted by the discount rate.

Since items' lifetimes follow negative exponential distributions, the probability of an item, say i, surviving $\Gamma(B)$ units of time is $e^{-\theta_i\Gamma(B)}$. To discount the sum of the several optimal expected cost components, i.e., the several $V(G)$, $G \subset \mathrm{B}$ such that $G = \left(i_{G_1}, \dots, i_{G_m}\right)$, we simply multiply them by $e^{-\delta\Gamma(B)}$. Hence, the costs incurred after the cycle when m $(1 \le m < b)$ items survive are given by Eq. (3) where $B \setminus G$ corresponds to the complement of G in B and, for each m, $\mathrm{Z}_m(B)$ is the sum of $\binom{b}{m} = \frac{b!}{m!(b-m)!}$ components.

$$\mathrm{Z}_m(B) = \sum_{\substack{G\subset \mathrm{B}\\ card(G)=m}} \left(V(G)e^{-\left(\theta_{i_{G_1}}+\cdots+\theta_{i_{G_m}}\right)\Gamma(B)} \prod_{j\in B\setminus G}\left(1-e^{-\theta_j\Gamma(B)}\right)\right) \tag{3}$$

when $G = B$, which is related to the case where all the items of B do survive the cycle of $\Gamma(B)$ units of time, the expression $\mathrm{Z}_b(B)$ corresponds precisely to the multiplication of $V(B)$ by the probability of having all the items of B surviving the cycle, i.e., $\mathrm{Z}_b(B)$ is given by Eq. (4).

$$\mathrm{Z}_b(B) = V(B)\prod_{i\in \mathrm{B}} e^{-\theta_i\Gamma(B)} = V(B)e^{-\left(\theta_{i_1}+\cdots+\theta_{i_b}\right)\Gamma(B)} \tag{4}$$

Then, the general expression of the total expected discounted cost, associated to the items i that do belong to B, is given by:

$$V(B) = \mathrm{Z}_0(B) + e^{-\delta\Gamma(B)}\sum_{m=1}^{b}\mathrm{Z}_m(B) \Leftrightarrow V(B) = \mathrm{Z}_0(B) + e^{-\delta\Gamma(B)}\sum_{m=1}^{b-1}\mathrm{Z}_m(B) + e^{-\delta\Gamma(B)}\mathrm{Z}_b(B) \Leftrightarrow V(B) - e^{-\delta\Gamma(B)}\mathrm{Z}_b(B)$$
$$= \mathrm{Z}_0(B) + e^{-\delta\Gamma(B)}\sum_{m=1}^{b-1}\mathrm{Z}_m(B)$$

Now, by applying Eq. (4) we have:

$$V(B) - e^{-\delta\Gamma(B)}\left(V(B)e^{-\left(\theta_{i_1}+\cdots+\theta_{i_b}\right)\Gamma(B)}\right) = \mathrm{Z}_0(B) + e^{-\delta\Gamma(B)}\sum_{m=1}^{b-1}\mathrm{Z}_m(B) \Leftrightarrow$$

$$V(B) = \frac{\mathrm{Z}_0(B) + e^{-\delta\Gamma(B)}\sum_{m=1}^{b-1}\mathrm{Z}_m(B)}{\left(1-e^{-\left(\delta+\theta_{i_1}+\cdots+\theta_{i_b}\right)\Gamma(B)}\right)} \tag{5}$$

By considering Eq. (2), expression (5) can also be rewritten as:

$$V(B) = \frac{C_o(B) + C_a(B) + C_h(B) + e^{-\delta\Gamma(B)}\sum_{m=1}^{b-1}\mathrm{Z}_m(B)}{\left(1-e^{-\left(\delta+\theta_{i_1}+\cdots+\theta_{i_b}\right)\Gamma(B)}\right)} \tag{6}$$

When $B = S$, where $card(S) = N$, Eq. (6) can be expressed as Eq. (7).

$$V(S) = \frac{C_o(S) + C_a(S) + C_h(S) + e^{-\delta T}\sum_{m=1}^{N-1}\mathrm{Z}_m(S)}{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)} \tag{7}$$

3.2 *Expression of the objective function with the integer multipliers*

In Subsection 0 we obtained $V(S)$ in Eq. (7), involving all the N types of items, from the expression derived for a subset B of S. An analogous development could be done to derive the corresponding expression with the integer multipliers. However, for the sake of simpler formulas, and without loss of generality, we derive the new model by directly considering the set S instead of a subset B. Hence, for the sake of convenience, we assume, in this Subsection 3.2, $B \subset S$ such that $card(B) = m = b$ and $B = (i_1, i_2, \dots, i_b)$.

The expression (7) of the JRP obsolescence model without the integer multipliers was derived by considering that the setup, acquisition and holding costs, respectively $C_o(S)$, $C_a(S)$ and $C_h(S)$, are strictly incurred during the first cycle. Moreover, all the other components of Eq. (7), namely expressed in Eq. (3) and Eq. (4), have been also obtained by considering the correspondent cost components as strictly incurred after the first cycle. However, the model is recursive, meaning that for computing the cost after the first cycle, another 'first' cycle is considered but only for the items that survived.

Extending our model by integrating the integer multipliers k_i, such that $T_i = k_i T$, $i = 1, \dots, N$, introduces a source of complexity in our model, particularly because the setup, acquisition and holding costs of the items, considered to be incurred during the first cycle, depend on the time interval between successive replenishments of each specific item. Indeed, when the integer multipliers are considered, the setup, acquisition and holding costs of an item i must be incurred during cycles of $T_i = k_i T$ units of time, instead of T units of time. This means that some items are not necessarily ordered in all cycles of T units of time. Thus, we need to define new expressions for the setup, acquisition and holding costs, where these costs are incurred during the first cycle of T units of time, in the context of extending Eq. (7) into a new JRP obsolescence model with the integer multipliers. To this purpose, a new expression to the costs incurred when obsolescence occurs after the first cycle, such as the ones derived in Eq. (3) and Eq. (4), also needs to be established.

Let us consider the additional notation for the extended model:

$k(B)$ – set of integer multipliers such that if $B = (i_1, i_2, \dots, i_b)$ then $k(B) = \left(k_{i_1}, k_{i_2}, \dots, k_{i_b}\right)$, with $T_{i_r} = k_{i_r} T$, $r = 1, \dots, b$;

T_i – time interval between successive replenishments of item i;

k_i − integer multiplier such that $T_i = k_i T$, $i = 1, \dots, N$;

One should note that optimizing the total expected discounted infinite horizon cost of S implies not only the determination of the optimal value of optimal value of T, as happens with the optimization of $V(S)$, but also the optimal values of $k(S)$, i.e., the optimal values of k_i such that $T_i = k_i T$, $i = 1, \dots, N$.

Before deriving the formulas of the ordering, acquisition, and holding costs during the cycle as well as the costs incurred when obsolescence occurs after the cycle, note that (7) is the sum of each of these four cost components divided by the denominator $\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)$. Each of these divisions can be interpreted as a discounted cost contribution to the value of $V(S)$, where the discount rate is equal to the sum of the discount rate (δ) with the sum of the obsolescence rates of the N items. For instance, the discounted ordering costs contribution to $V(S)$ in (7) is given by $\frac{C_o(S)}{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}$. The reason is due to the fact that (proof in Appendix A):

$$\frac{C_o(S)}{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)} = \sum_{u=0}^{\infty} C_o(S) e^{-(\delta+\theta_1+\cdots+\theta_N)uT} \tag{8}$$

In other words, by Eq. (8), we can say that $\frac{C_o(S)}{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}$ is the present value of a perpetuity with a cash flow of $C_o(S)$ occurring at every T units of time and a continuous discount rate of $(\delta + \theta_1 + \cdots + \theta_N)$. This is also the case for the other cost components enumerated supra. However, as referred above, an issue is raised when, by introducing the integer multipliers, formulations for these cost components must be obtained when successive replenishments of an item i occur only once in every $T_i = k_i T$ units of time. Below we derive the formulas of the four cost components above in order to have the corresponding costs considered during T units of time instead of T_i units of time.

Setup costs

The value of the setup costs incurred during the cycle, considering that some items are not ordered in all cycles, say $\tilde{C}_o(S)$, contains two components, namely one with respect to the major setup cost A, which is incurred in every period between successive replenishments, and another one with respect to the minor setup costs of the items ordered in a replenishment such that if an item i is ordered, the minor setup cost a_i is incurred only once in every $T_i = k_i T$ units of time, and not once in every period of T units of time.

This means that the cost contribution of the major setup cost during the cycle to $\tilde{C}_o(S)$ is A, but the minor cost contribution of an item i needs an adjustment in order to be incurred during T units of time instead of T_i units of time.

Property for adjusting the minor setup cost

A minor setup cost a_i incurred every k_iT units of time is equivalent to a minor setup cost $\tilde{a}_i$ incurred every T units of time, with:

$$\tilde{a}_i = a_i \frac{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)k_iT}\right)} \tag{9}$$

Proof:

The infinite sum of discounted values of the minor setup cost, $\tilde{a}_i$, at the rate $(\delta + \theta_1 + \cdots + \theta_N)$, considering T units of time is:

$$\sum_{u=0}^{\infty} \tilde{a}_i e^{-(\delta+\theta_1+\cdots+\theta_N)uT}$$

By applying Eq. (8) we get:

$$\sum_{u=0}^{\infty} \tilde{a}_i e^{-(\delta+\theta_1+\cdots+\theta_N)uT} = \frac{\tilde{a}_i}{\left(1 - e^{-(\delta+\theta_1+\ldots+\theta_N)T}\right)} \tag{10}$$

Thus, dividing the minor setup cost $\tilde{a}_i$ by $\left(1 - e^{-(\delta+\theta_1+\ldots+\theta_N)T}\right)$ is the same as considering an infinite sum where $\tilde{a}_i$ is discounted at the beginning of each cycle.

If instead we discount a minor setup cost a_i at the beginning of each k_i cycles, that is, at periods of $T_i = k_iT$, we would get the following present value of the minor setup costs:

$$\sum_{u=0}^{\infty} a_i e^{-(\delta+\theta_1+\cdots+\theta_N)uk_iT}$$

Following the same reasoning used in (10), we would get:

$$\sum_{u=0}^{\infty} a_i e^{-(\delta+\theta_1+\cdots+\theta_N)uk_iT} = \frac{a_i}{\left(1 - e^{-(\delta+\theta_1+\ldots+\theta_N)k_iT}\right)} \tag{11}$$

It can be seen that the infinite summations in Eq. (10) and Eq. (11) are identical if:

$$\frac{\tilde{a}_i}{\left(1 - e^{-(\delta+\theta_1+\ldots+\theta_N)T}\right)} = \frac{a_i}{\left(1 - e^{-(\delta+\theta_1+\ldots+\theta_N)k_iT}\right)}$$

Thus,

$$\tilde{a}_i = a_i \frac{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)k_iT}\right)}$$

as we wanted to prove.

So, the formulation of $\tilde{C}_o(S)$ when the N items are considered is given by (12):

$$\tilde{C}_o(S) = A + \sum_{i=1}^{N} a_i \frac{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)k_iT}\right)} \tag{12}$$

Acquisition costs

The acquisition costs incurred during the cycle, considering that some items are not ordered in all cycles, say $\tilde{C}_a(S)$, have the same issue as the minor setup costs discussed above. If an item i is ordered, the acquisition costs are determined by considering the unit cost c_i and the number of items needed to satisfy the demand for $T_i = k_iT$ units of time. The quantity of items needed to satisfy the demand for k_i multiples of the cycle is $D_iT_i = D_ik_iT$, meaning that the acquisition costs of a replenishment that includes the item i is given by $c_iD_iT_i = c_iD_ik_iT$. In this context, an adjustment is required to have the acquisition costs incurred during T, instead of T_i units of time.

Property for adjusting the acquisition costs

The acquisition costs $c_iD_ik_iT$ incurred every k_iT units of time are equivalent to the acquisition costs $\tilde{c}_i$ incurred every T units of time, with:

$$\tilde{c}_i = c_iD_ik_iT \frac{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)k_iT}\right)} \tag{13}$$

Proof:

The proof is analogous to the one considered for the minor setup cost adjustment. Thus, replacing $\tilde{a}_i$ by $\tilde{c}_i$, a_i by $c_iD_ik_iT$ and following the same steps of the proof used above to obtain $\tilde{a}_i$, Eq. (13) holds and the formulation of $\tilde{C}_a(S)$ when the N items are considered is given by Eq. (14).

$$\tilde{C}_a(S) = \sum_{i=1}^{N} c_iD_ik_iT \frac{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)k_iT}\right)} \tag{14}$$

Holding costs

The formulation of $\tilde{C}_h(S)$ is obtained by following the same reasoning used to prove the formulation of $\tilde{C}_o$.

Property for adjusting the holding costs

Considering by (1) that $H_i(T_i)$ expresses the expected holding costs of an item i during T_i units of time, the expected holding costs of the item i during the cycle of T units of time, say $\widetilde{H}_i(T)$, is given by:

$$\widetilde{H}_i(T) = H_i(k_iT) \frac{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)k_iT}\right)} \tag{15}$$

Proof:

Replacing $\tilde{a}_i$ by $\widetilde{H}_i(T)$, a_i by $H_i(k_iT)$ and following the same steps of the proof used above to obtain $\tilde{a}_i$, (15) holds and the formulation of $\tilde{C}_h(S)$ is given by Eq. (16).

$$\tilde{C}_h(S) = \sum_{i=1}^{N} H_i(k_iT) \frac{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)k_iT}\right)} \tag{16}$$

After demonstrating the formulations of the setup costs Eq. (12), acquisition costs Eq. (14) and holding costs Eq. (16),the costs incurred when obsolescence occurs during the cycle of T units of time, say $\tilde{Z}_0(S) = \tilde{C}_o(S) + \tilde{C}_a(S) + \tilde{C}_h(S)$, is given by Eq. (17), which is simplified in Eq. (18).

$$\begin{aligned} \tilde{Z}_0(S) &= \tilde{C}_o(S) + \tilde{C}_a(S) + \tilde{C}_h(S) \\ &= A + \sum_{i=1}^{N} a_i \frac{\left(1-e^{-(\delta+\theta_1+\ldots+\theta_N)T}\right)}{\left(1-e^{-(\delta+\theta_1+\ldots+\theta_N)k_iT}\right)} + \sum_{i=1}^{N} c_iD_ik_iT \frac{\left(1-e^{-(\delta+\theta_1+\ldots+\theta_N)T}\right)}{\left(1-e^{-(\delta+\theta_1+\ldots+\theta_N)k_iT}\right)} \\ &\quad + \sum_{i=1}^{N} H_i(k_iT) \frac{\left(1-e^{-(\delta+\theta_1+\ldots+\theta_N)T}\right)}{\left(1-e^{-(\delta+\theta_1+\ldots+\theta_N)k_iT}\right)} \end{aligned} \tag{17}$$

$$\tilde{Z}_0(S) = A + \left(1-e^{-(\delta+\theta_1+\ldots+\theta_N)T}\right) \sum_{i=1}^{N} \frac{a_i + c_iD_ik_iT + H_i(k_iT)}{\left(1-e^{-(\delta+\theta_1+\ldots+\theta_N)k_iT}\right)} \tag{18}$$

Total costs

In Subsection 0, these costs were derived in Eq. (3) and Eq. (4) by considering the multiplication of probabilities of survival and the values of $V(B)$ over all possible combinations where at least one item belonging to the subset B of S survived. By analyzing the factors in Eq. (3) and Eq. (4), the probabilities of survival do not need an adjustment through the introduction of the integer multipliers because they are already calculated during the cycle of T units of time. The same can be said not only for the case where $B = S$, but also for any subset B as, in this case, the probabilities must be simply calculated during $\Gamma(B)$ units of time. However, an adjustment needs to be made to the values of the total expected discounted costs associated to the items set under analysis. This is because these values are calculated in Eq. (6) and Eq. (7) by considering the setup, the acquisition and the holding costs without the impact of the integer multipliers. However, to integrate the impact of the integer multipliers we have to replace these costs in $V(B)$ by the formulas (12), (14) and (16), originating in this manner a modified $V(B)$, say $\tilde{V}(B)$. Thus, the expression of the costs incurred when obsolescence of at least one item occurs after the cycle, $\tilde{Z}_m(S)$, can be expressed by Eq. (19).

$$\tilde{Z}_m(S) = \sum_{\substack{B \subset S \\ card(B)=m}} \left(\tilde{V}(B) e^{-(\theta_{i_1}+\cdots+\theta_{i_m})\mathrm{T}} \prod_{j \in S \setminus B} \left(1 - e^{-\theta_j \mathrm{T}}\right) \right) \tag{19}$$

The expression of the case where all the N items survive the cycle of T units of time, $\tilde{Z}_N(S)$, is given by (20). The value of $\tilde{V}(S)$ corresponds to the value of $V(S)$ where the costs in Eq. (2) and Eq. (3) are replaced by the costs in Eq. (18) and Eq. (19).

$$\tilde{Z}_N(S) = \tilde{V}(S) \prod_{i=1}^{N} e^{-\theta_i T} = \tilde{V}(S) e^{-(\theta_1+\cdots+\theta_N)T} \tag{20}$$

Please note that despite these formulas having been derived in this Subsection 3.2 with the focus on the case where $B = S$, they can be easily established for any nonempty subset B of S.

Now that we have developed the expressions of the cost components with the integer multipliers of our model, the total expected discounted infinite horizon cost associated to the N items that do belong to S, $\tilde{V}(S)$, is given by Eq. (21).

$$\tilde{V}(S) = \frac{\tilde{Z}_0(S) + e^{-\delta T} \sum_{m=1}^{N-1} \tilde{Z}_m(S)}{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)} \tag{21}$$

By considering Eq. (18) in Eq. (21), we finally obtain Eq. (22).

$$\tilde{V}(S) = \frac{A + e^{-\delta T} \sum_{m=1}^{N-1} \tilde{Z}_m(S)}{\left(1 - e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)} + \sum_{i=1}^{N} \frac{a_i + c_i D_i k_i T + H_i(k_i T)}{\left(1 - e^{-(\delta+\theta_1+\ldots+\theta_N)k_i T}\right)} \tag{22}$$

In summary, the model with the integer multipliers expressed in (22) determines the total expected discounted infinite horizon cost associated to N items that are subject to obsolescence. Subsection 3.3 describes a recursive procedure that intends to find the optimal parameters of the (T, k_i) ordering policy that minimizes (22).

3.3 A procedure to optimize the objective function

In the same way as proposed by Afonso et al. (2022) for the approximate model, to compute the minimum of Eq. (22), it is necessary to consider a recursive procedure to determine the several $\tilde{Z}_m(S)$ $(m = 1,2,\ldots,N-1)$ that appear in Eq. (19). During the first iteration, we determine $\tilde{Z}_1(S)$ by obtaining the values corresponding to the cases where just one of the items survive the cycle $\Gamma\big((i)\big)$ $(i = 1, 2, \ldots, N)$. The calculation of the N values of $\tilde{V}\big((i)\big)$ needed to calculate $\tilde{Z}_1(S)$ are obtained by optimizing the single-item model developed by van Delft & Vial (1996) and described by Afonso et al. (2022), under the JRP context. In the second iteration, the optimal single-item values $\tilde{V}\big((i)\big)$, determined in the first iteration to calculate $\tilde{Z}_1(S)$, are processed as an input to calculate the $\tilde{Z}_2(S)$, which must consider all the cases where two of the items survive the cycle $\Gamma\big((i_1, i_2)\big)$. That is to say that the optimal values of $\tilde{V}\big((i_1)\big)$ and $\tilde{V}\big((i_2)\big)$ calculated in the first iteration are used as constants to compute the optimal values $\tilde{V}\big((i_1, i_2)\big)$ in the second iteration. Note that in this second iteration, we must compute optimal values for all subsets B of S involving two of the N items. The number of subsets B in this iteration is $\binom{N}{2} = \frac{N!}{2!(N-2)!}$. All these expected discounted costs involving any two items are then used to calculate $\tilde{Z}_2(S)$. Hence, the procedure continues in this way until the last iteration (i.e., by computing successively $\tilde{Z}_3(S)$, $\tilde{Z}_4(S)$, …, $\tilde{Z}_{N-1}(S)$), in which the optimal value of $\tilde{V}(S)$ is finally computed.

4. Numerical results and discussion

The discounted JRP obsolescence precise model proposed in Subsection 3.2 was implemented and tested with the case bases depicted in Table 1.These test cases were used to perform sensitivity analyses on underlying parameters, in order to highlight the impact of the core features of the model, such as the obsolescence, time value of money and JRP setup costs, on the ordering policies. The base cases I and II of Table 1 are also used in Afonso et al. (2022).

A group of tests is constituted by six or seven test cases. The number of test cases in each group depends on the parameters where sensitivity analyses are performed. The definition of a group of tests considers: 1) the base case used to perform the test cases, 2) the selected parameters used to perform the sensitivity analyses, 3) specific values of other parameters where sensitivity analysis is not being performed but whose values are different from the parameters of the used base case. For instance, if we select the parameters depicted in base case I of Table 1 to perform sensitivity analysis on the discount rate, using the (additional) values 0.1 and 0.2, and a change on the major setup cost, which is set to 1000, this means that we consider three test cases where all the parameters are equal to the base case I, except the values of the discount rate which are progressively changed from 0.05 (depicted in Table 1) for test case 1, 0.1 for test case 2 and 0.2 for test case 3, and the value of the major setup cost is set to 1000 for all the three test cases (instead of 100, which is the value of the major setup cost of the base case I).

Table 1
Figures of the base cases

Parameter	Base case I	Base case II	Base case III	Base case IV
δ	0.05	0.1	0.05	0.05
$\theta_1, \theta_2\ \theta_3$	0.2	0.3	0.1	0.1
A (\$)	100	1000	1000	1000
a_1 (\$)	10	50	950	950
a_2 (\$)	15	70	1000	940
a_3 (\$)	10	70	1200	2000
D_1 (units)	80	150	150	150
D_2 (units)	200	400	900	140
D_3 (units)	60	400	2000	1500
c_1 (\$)	2	4	4	4
c_2 (\$)	3	8	15	5
c_3 (\$)	1	10	5	25
h_1 (\$/unit/time)	0.2	0.6	0.6	0.6
h_2 (\$/unit/time)	0.3	1.2	1	0.5
h_3 (\$/unit/time)	0.1	1.5	1.5	3

Afonso et al. (2022) performed five groups of tests, that we now repeat, each of them containing six test cases, except the fifth group which contains seven test cases:

1. The first group includes six test cases based on base case I of Table 1. The test cases 1, 2 and 3 perform sensitivity analysis on the obsolescence rates with the values 0.2, 0.3 and 0.5. The same is done for the last three test cases. The discount rate is 0.1 for all the test cases. The major setup cost is 1000 in test cases 1, 2 and 3, and is equal to zero in test cases 4, 5 and 6.

2. The second group uses the base case I and includes six test cases as well. The sensitivity analysis is performed on the major setup cost with the values 100, 1000 and 10000. The obsolescence rates of the items are all set to 0.5 in the test cases 1, 2 and 3. The test cases 4, 5 and 6 are equal to test cases 1, 2 and 3, respectively, except for the obsolescence rates which are equal to 0.2, 0.3 and 0.5 for items 1, 2 and 3, respectively.

3. The test cases of the third group are equal to the test cases of the second group, except for differences on the setup costs. The major setup cost is the same for all test cases (equal to 1000). The test cases 1, 2 and 3 perform sensitivity analysis on the minor setup cost of item 1 with the values 10, 50 and 3000, while the test cases 4, 5 and 6 perform sensitivity analysis on the minor setup cost of item 3 with the values 10, 70 and 500, respectively.

4. The fourth group of tests is based on the base case II of Table 1. The test cases 1, 2 and 3 are equal, except for the progressive increase of the discount rate, which takes the values 0.05, 0.1 and 0.2. The obsolescence rates have different values from item to item (0.2, 0.3 and 0.5, respectively). The test cases 4, 5 and 6 are equal to the first three test cases, but the items are not subject to obsolescence, i.e., the obsolescence rates are zero.

5. The fifth group includes seven test cases, it is based on the base case II and takes the test case 1 as a basis for the sensitivity analyses on the following parameters: holding cost, demand and unit cost. Therefore, test cases 1, 2 and 3 are equal, except for the progressive increase of the holding cost parameter (0.2, 0.6 and 1.5 for item 1; 0.3, 1.2 and 3 for item 2; 0.1, 1.5 and 7.5 for item 3), while the unit cost and demand parameters consider the figures of the base case I; test cases 1, 4 and 5 are equal, except for the progressive increase of the demand parameter (80, 150 and 300 for item 1; 200, 400 and 600 for item 2; 60, 400 and 1200 for item 3), while the holding cost and unit cost parameters consider the figures of the base case I; and test cases 1, 6 and 7 are equal, except for the progressive increase of the item unit cost parameter (2, 4 and 6 for item 1; 3, 8 and 12 for item 2; 1, 10 and 30 for item 3), while the holding cost and demand parameters consider the figures of the base case I.

Table 2 compares the numerical results obtained from executing these five groups of test cases with the approximate model and the precise model. We use the expression “Approximate model” to identify the approach proposed by Afonso et al. (2022) and “Precise model” to identify the approach proposed in this paper, since this approach avoids the approximations used by Afonso et al. (2022) regarding the inclusion of the k_is. For each of the five group of tests, it is identified the corresponding base case, the parameters with different values from the ones of this base case and, finally, optimal values obtained for the cycle length (T) and the integer multipliers (k_i), and the corresponding order quantities ($Q_i = D_i k_i T$). The columns correspond to the several test cases. As explained above, column 7 only has figures on the fifth group because this group is the only one with seven test cases. All the other groups of tests have six test cases.

Table 2
Numerical results of the approximate and the precise models

Approximate model

Test case		1	2	3	4	5	6	7
Base case I, Group of tests 1	**Changes on parameters of the base case**							
	$\theta 1$	0.2	0.3	0.5	0.2	0.3	0.5	
	$\theta 2$	0.2	0.3	0.5	0.2	0.3	0.5	
	$\theta 3$	0.2	0.3	0.5	0.2	0.3	0.5	
	A	1000	1000	1000	0	0	0	
	Variables							
	T	2.73	2.38	1.83	0.50	0.43	0.35	-
	k1	1	1	1	1	1	1	-
	k2	1	1	1	1	1	1	-
	k3	1	1	2	1	1	1	-
	Q1	218.24	190.28	146.28	40.21	34.54	27.89	-
	Q2	545.60	475.70	365.69	100.53	86.36	69.73	-
	Q3	163.68	142.71	219.41	30.16	25.91	20.92	-
Base case I, Group of tests 2	**Changes on parameters of the base case**							
	A	100	1000	10000	100	1000	10000	-
	$\theta 1$	0.5	0.5	0.5	0.2	0.2	0.2	-
	$\theta 2$	0.5	0.5	0.5	0.3	0.3	0.3	-
	$\theta 3$	0.5	0.5	0.5	0.5	0.5	0.5	-
	Variables							
	T	0.67	1.93	4.86	0.92	2.24	6.15	-
	k1	1	1	1	1	2	2	-
	k2	1	1	1	1	1	1	-
	k3	2	2	2	1	1	1	-
	Q1	53.46	154.05	389.16	73.86	358.67	984.27	-
	Q2	133.65	385.13	972.91	184.65	448.34	1230.34	-
	Q3	80.19	231.08	583.75	55.39	134.50	369.10	-
Base case I, Group of tests 3	**Changes on parameters of the base case**							
	a1	10	50	3000	10	10	10	-
	a3	10	10	10	10	70	500	-
	A	1000	1000	1000	1000	1000	1000	-
	$\theta 1$	0.5	0.5	0.5	0.2	0.2	0.2	-
	$\theta 2$	0.5	0.5	0.5	0.3	0.3	0.3	-
	$\theta 3$	0.5	0.5	0.5	0.5	0.5	0.5	-
	Variables							
	T	1.93	1.63	1.77	2.24	2.07	2.20	-
	k1	1	2	3	2	2	2	-
	k2	1	1	1	1	1	1	-
	k3	2	2	2	1	2	2	-
	Q1	154.05	260.25	424.36	358.67	331.81	351.95	-
	Q2	385.13	325.31	353.64	448.34	414.76	439.94	-
	Q3	231.08	195.19	212.18	134.50	248.86	263.96	-
Base case II, Group of tests 4	**Changes on parameters of the base case**							
	δ	0.05	0.1	0.2	0.05	0.1	0.2	-
	$\theta 1$	0.2	0.2	0.2	0	0	0	-
	$\theta 2$	0.3	0.3	0.3	0	0	0	-
	$\theta 3$	0.5	0.5	0.5	0	0	0	-
	Variables							
	T	0.74	0.74	0.67	2.42	1.70	1.19	-
	k1	2	1	1	1	1	1	-
	k2	1	1	1	1	1	1	-
	k3	1	1	1	1	1	1	-
	Q1	220.87	110.37	100.59	363.08	254.61	177.95	-
	Q2	294.50	294.32	268.24	968.21	678.95	474.54	-
	Q3	294.50	294.32	268.24	968.21	678.95	474.54	-
Base case II, Group of tests 5	**Changes on parameters of the base case**							
	h1	0.2	0.6	1.5	0.2	0.2	0.2	0.2
	h2	0.3	1.2	3	0.3	0.3	0.3	0.3
	h3	0.1	1.5	7.5	0.1	0.1	0.1	0.1
	D1	80	80	80	150	300	80	80
	D2	200	200	200	400	600	200	200
	D3	60	60	60	400	1200	60	60
	c1	2	2	2	2	2	4	6
	c2	3	3	3	3	3	8	12
	c3	1	1	1	1	1	10	30
	Variables							
	T	2.50	2.25	1.84	1.67	1.23	1.49	1.11
	k1	1	1	1	1	1	1	1
	k2	1	1	1	1	1	1	1
	k3	1	1	1	1	1	1	1
	Q1	200.04	180.28	146.83	251.15	367.65	119.27	88.57
	Q2	500.09	450.70	367.07	669.73	735.30	298.19	221.42
	Q3	150.03	135.21	110.12	669.73	1470.61	89.46	66.43

Precise model

Test case		1	2	3	4	5	6	7
Base case I, Group of tests 1	**Changes on parameters of the base case**							
	$\theta 1$	0.2	0.3	0.5	0.2	0.3	0.5	
	$\theta 2$	0.2	0.3	0.5	0.2	0.3	0.5	
	$\theta 3$	0.2	0.3	0.5	0.2	0.3	0.5	
	A	1000	1000	1000	0	0	0	
	Variables							
	T	2.73	2.39	1.98	0.50	0.43	0.35	-
	k1	1	1	1	1	1	1	-
	k2	1	1	1	1	1	1	-
	k3	1	1	1	1	1	1	-
	Q1	218.60	191.11	158.18	40.16	34.50	27.86	-
	Q2	546.49	477.79	395.45	100.41	86.25	69.65	-
	Q3	163.95	143.34	118.64	30.12	25.87	20.90	-
Base case I, Group of tests 2	**Changes on parameters of the base case**							
	A	100	1000	10000	100	1000	10000	-
	$\theta 1$	0.5	0.5	0.5	0.2	0.2	0.2	-
	$\theta 2$	0.5	0.5	0.5	0.3	0.3	0.3	-
	$\theta 3$	0.5	0.5	0.5	0.5	0.5	0.5	-
	Variables							
	T	0.73	2.09	5.46	0.92	2.61	7.39	-
	k1	1	1	1	1	1	1	-
	k2	1	1	1	1	1	1	-
	k3	1	1	1	1	1	1	-
	Q1	58.44	167.31	436.61	73.86	208.63	591.37	-
	Q2	146.10	418.28	1091.52	184.65	521.58	1478.43	-
	Q3	43.83	125.48	327.46	55.39	156.47	443.53	-
Base case I, Group of tests 3	**Changes on parameters of the base case**							
	a1	10	50	3000	10	10	10	-
	a3	10	10	10	10	70	500	-
	A	1000	1000	1000	1000	1000	1000	-
	$\theta 1$	0.5	0.5	0.5	0.2	0.2	0.2	-
	$\theta 2$	0.5	0.5	0.5	0.3	0.3	0.3	-
	$\theta 3$	0.5	0.5	0.5	0.5	0.5	0.5	-
	Variables							
	T	2.09	2.12	3.16	2.61	2.65	2.89	-
	k1	1	1	1	1	1	1	-
	k2	1	1	1	1	1	1	-
	k3	1	1	1	1	1	1	-
	Q1	167.31	169.26	252.81	208.63	211.75	231.04	-
	Q2	418.28	423.15	632.02	521.58	529.37	577.60	-
	Q3	125.48	126.95	189.61	156.47	158.81	173.28	-
Base case II, Group of tests 4	**Changes on parameters of the base case**							
	δ	0.05	0.1	0.2	0.05	0.1	0.2	-
	$\theta 1$	0.2	0.2	0.2	0	0	0	-
	$\theta 2$	0.3	0.3	0.3	0	0	0	-
	$\theta 3$	0.5	0.5	0.5	0	0	0	-
	Variables							
	T	0.78	0.74	0.67	2.42	1.70	1.19	-
	k1	1	1	1	1	1	1	-
	k2	1	1	1	1	1	1	-
	k3	1	1	1	1	1	1	-
	Q1	116.35	110.37	100.59	363.08	254.61	177.95	-
	Q2	310.26	294.32	268.24	968.21	678.95	474.54	-
	Q3	310.26	294.32	268.24	968.21	678.95	474.54	-
Base case II, Group of tests 5	**Changes on parameters of the base case**							
	h1	0.2	0.6	1.5	0.2	0.2	0.2	0.2
	h2	0.3	1.2	3	0.3	0.3	0.3	0.3
	h3	0.1	1.5	7.5	0.1	0.1	0.1	0.1
	D1	80	80	80	150	300	80	80
	D2	200	200	200	400	600	200	200
	D3	60	60	60	400	1200	60	60
	c1	2	2	2	2	2	4	6
	c2	3	3	3	3	3	8	12
	c3	1	1	1	1	1	10	30
	Variables							
	T	2.51	2.26	1.84	1.67	1.23	1.49	1.11
	k1	1	1	1	1	1	1	1
	k2	1	1	1	1	1	1	1
	k3	1	1	1	1	1	1	1
	Q1	200.95	180.48	146.83	251.15	367.65	119.27	88.57
	Q2	502.38	451.20	367.07	669.73	735.30	298.19	221.42
	Q3	150.71	135.36	110.12	669.73	1470.61	89.46	66.43

Roughly speaking, the results obtained with the precise model are aligned with the ones discussed by Afonso et al. (2022). The only exception occurs, to a certain extent, when we look at the results of the third group of tests in Table 2. When the approximate model is applied to this group of tests, a progressive increase of the minor setup costs (item 1 in the first three test cases and item 3 in the last three) leads to an increase of the respective integer multiplier. With the precise model we notice a different pattern, with the progressive increase in the minor setup cost causing an increase in the cycle length.

In addition to the abovementioned, when comparing the results obtained with the integer multipliers between the approximate and the precise models in Table 2, we can observe that the precise model is more "conservative" with respect to the integer multipliers. As can be seen, in all the five groups of tests, the optimal integer multipliers are equal to one.

Another aspect to highlight are the slight differences observed in the optimal cycle lengths and, consequently, ordered quantities of the groups of tests 1 and 5. In the first group of tests, despite the optimal integer multipliers being equal to one for both models (except for test case 3 where $k_3 = 2$ for the approximate model), the cycle lengths of the test cases are not exactly equal, as one would expect for situations where the optimal integer multiplier are all equal to one through execution of both models. The reason for having different cycle lengths is due to the fact that the integer multipliers of the intermediate iterations are not all equal to one. For example, during calculations we can verify that $k_3 = 2$ in the intermediate iteration 2 of the approximate model, involving the items 2 and 3 of the test case 2. On the fifth group of tests, comparing the results obtained with the precise and approximate models in test cases 3, 4, 5, 6 and 7, we can verify that both the optimal cycle lengths and ordered quantities are all equal. The reason has to do with the fact that the optimal integer multipliers are all equal to one, not only in the last (third) iteration, but also in the second iteration (one should note that in the first iteration the integer multipliers do not apply because we are dealing with single-item subproblems).

In the light of characteristics of the precise model, we executed additional five groups of tests, based on the base cases III and IV of Table 1, in order to perform sensitivity analyses on the parameters, and to promote the occurrence of integer multipliers greater than one. All these groups of tests contain six test cases, except the eighth group which has seven test cases. In this scope, the following groups of tests (from group of tests 6 to group of tests 10) were performed, where the first three are based on base case III and the last two are based on the case base IV:

6. As far as the sixth group of tests is concerned, the test cases 8, 9 and 10 increase progressively the obsolescence rates of the three items to 0.02, 0.1 and 0.3. In the test cases 11, 12 and 13 the obsolescence rates of the items are set to 0.1 (according to the base case III) and the discount rate increases to 0.03, 0.05 and 0.1, respectively.

7. Considering the seventh group of tests, the differences among the test cases occur on the setup costs: test cases 8, 9 and 10 differ on the progressive increase of the major setup cost to 100, 1000, 10000, respectively; while in test cases 11, 12 and 13 the differences occur in the minor setup cost of item 1, which is successively increased to 90, 950 and 9000.

8. The eighth group of tests performs sensitivity analyses on three parameters: holding cost, demand and unit cost. The test case 8 is used as the first test case to conduct the sensitivity analyses across the three parameters. Thereby, test cases 8, 9 and 10 are used to perform the analysis on the holding cost (by increasing, respectively, the values of the item 1 to 0.2, 0.6 and 0.7; the values of the item 2 to 0.3, 1 and 3; and the values of the item 3 to 0.1, 1.5 and 7.5); test cases 8, 11 and 12 are used to perform the analysis on demand (by increasing respectively the values of the item 1 to 80, 150 and 300; the values of the item 2 to 500, 900 and 3000; and the values of the item 3 to 900, 2000 and 4000); and, finally, test cases 8, 13 and 14 are used to the analysis on the unit cost (by increasing respectively the values of the item 1 to 2, 4 and 6; the values of the item 2 to 10, 15 and 20; and the values of the item 3 to 2, 5 and 8). Thus, the progressive increase of the values of the three parameters is done simultaneously over the three items.

9. The ninth group of tests performs sensitivity analyses on the same parameters of the sixth group of tests and with the same figures. The only difference here relies on the use of the base case IV.

10. The sensitivity analyses of the tenth group of tests are performed on the setup costs with the same figures of the seventh group of tests above, but based on the base case IV.

The results obtained by executing these groups of tests are presented in Table 3, which depicts the input parameters changes on correspondent base case figures, for performing the sensitivity analyses, and exhibits the obtained optimal values, as well.

The results illustrated by the group of tests 6 of Table 3 allow us to conclude that the progressive increase of the obsolescence and financial risks, through the obsolescence rates and discount rate, respectively, implies a progressive decrease of the period between successive replenishments and, consequently, a decrease of the ordered quantities. These results are aligned with the results obtained by van Delft & Vial (1996) with the single-item model and by Afonso et al. (2022) with the approximate model as well.

Based on the group of tests 7, we conclude that the progressive increase of the setup costs causes a progressive increase of the period between successive replenishments and, as one would expect, the ordered quantities. The exception in the ordered quantities is justified by the integer multipliers. The ordered quantities from test cases 9 to 10 decrease from 383.92 to 375.69 but the optimal k_1 decreases from 2 to 1, implying that item 1 is ordered at every cycle in test case 10. The increase of the minor setup cost of item 1 in test cases 11, 12 and 13 implies a progressive increase of the multiplier k_1 to 1, 2 and 4, respectively, which justifies a relevant increase of the ordered quantities of item 1 from 187.73 to 842.17 units over test cases 11, 12 and 13.

Table 3
Numerical results of the test cases based on case bases III and IV

Test case		8	9	10	11	12	13	14
Base case III Group of tests 6	**Changes on parameters of the base case**							
	$\theta 1$	0.02	0.1	0.3	0.1	0.1	0.1	-
	$\theta 2$	0.02	0.1	0.3	0.1	0.1	0.1	-
	$\theta 3$	0.02	0.1	0.3	0.1	0.1	0.1	-
	δ	0.05	0.05	0.05	0.03	0.05	0.1	-
	Variables							
	T	1.87	1.28	0.81	1.37	1.28	1.12	-
	k1	3	2	2	2	2	2	-
	k2	1	1	1	1	1	1	-
	k3	1	1	1	1	1	1	-
	Q1	843.18	383.92	244.33	409.68	383.92	335.64	-
	Q2	1686.36	1151.77	733.00	1229.05	1151.77	1006.93	-
	Q3	3747.46	2559.48	1628.89	2731.23	2559.48	2237.62	-
	Costs							
	$\widetilde{V}(S)$ ($)	401125.48	202952.48	98406.27	231176.63	202952.48	156682.87	-
Base case III Group of tests 7	**Changes on parameters of the base case**							
	A	100	1000	10000	1000	1000	1000	-
	a1	950	950	950	90	950	9000	-
	Variables							
	T	1.11	1.28	2.50	1.25	1.28	1.40	-
	k1	2	2	1	1	2	4	-
	k2	1	1	1	1	1	1	-
	k3	1	1	1	1	1	1	-
	Q1	332.56	383.92	375.69	187.73	383.92	842.17	-
	Q2	997.67	1151.77	2254.16	1126.40	1151.77	1263.25	-
	Q3	2217.05	2559.48	5009.25	2503.11	2559.48	2807.22	-
	Costs							
	$\widetilde{V}(S)$ ($)	195261.86	202952.48	250429.26	199741.18	202952.48	220798.28	-
Base case III Group of tests 8	**Changes on parameters of the base case**							
	h1	0.2	0.6	0.7	0.2	0.2	0.2	0.2
	h2	0.3	1	3	0.3	0.3	0.3	0.3
	h3	0.1	1.5	7.5	0.1	0.1	0.1	0.1
	D1	80	80	80	150	300	80	80
	D2	500	500	500	900	3000	500	500
	D3	900	900	900	2000	4000	900	900
	c1	2	2	2	2	2	4	6
	c2	10	10	10	10	10	15	20
	c3	2	2	2	2	2	5	8
	Variables							
	T	2.41	2.26	1.87	1.77	1.07	1.84	1.55
	k1	2	2	2	2	2	2	2
	k2	1	1	1	1	1	1	1
	k3	1	1	1	1	1	1	1
	Q1	386.04	361.79	299.11	531.81	639.29	294.97	248.10
	Q2	1206.37	1130.60	934.72	1595.44	3196.45	921.78	775.31
	Q3	2171.46	2035.07	1682.49	3545.42	4261.93	1659.21	1395.56
	Costs							
	$\widetilde{V}(S)$ ($)	69621.91	71179.59	76074.15	119573.45	306598.31	112067.14	153095.68
Base case IV Group of tests 9	**Changes on parameters of the base case**							
	$\theta 1$	0.02	0.1	0.3	0.1	0.1	0.1	-
	$\theta 2$	0.02	0.1	0.3	0.1	0.1	0.1	-
	$\theta 3$	0.02	0.1	0.3	0.1	0.1	0.1	-
	δ	0.05	0.05	0.05	0.03	0.05	0.1	-
	Variables							
	T	1.49	1.00	0.66	1.12	1.00	0.85	-
	k1	3	3	2	2	3	3	-
	k2	3	2	2	2	2	3	-
	k3	1	1	1	1	1	1	-
	Q1	671.93	451.92	198.77	334.66	451.92	380.68	-
	Q2	627.13	281.20	185.52	312.35	281.20	355.30	-
	Q3	2239.76	1506.41	993.87	1673.31	1506.41	1268.94	-
	Costs							
	$\widetilde{V}(S)$ ($)	625981.47	311072.96	147201.71	354903.44	311072.96	239195.16	
Base case IV Group of tests 10	**Changes on parameters of the base case**							
	A	100	1000	10000	1000	1000	1000	-
	a1	950	950	950	90	950	9000	-
	Variables							
	T	0.84	1.00	2.02	1.02	1.00	1.04	-
	k1	3	3	1	1	3	6	-
	k2	3	2	1	2	2	2	-
	k3	1	1	1	1	1	1	-
	Q1	376.52	451.92	303.00	152.30	451.92	933.54	-
	Q2	351.42	281.20	282.80	284.30	281.20	290.43	-
	Q3	1255.08	1506.41	3030.00	1523.02	1506.41	1555.90	-
	Costs							
	$\widetilde{V}(S)$ ($)	302947.68	311072.96	363386.48	307964.70	311072.96	328507.95	-

According to the results illustrated by the group of tests 8, the period between successive replenishments decreases progressively with the progressive increase of the holding cost, demand, and unit cost. A difference can be pointed out with the ordered quantities, which also decrease with the increase of the holding costs and the unit costs but increase with the increase of the demand.

The results of the groups of tests 9 and 10 of Table 3 allow us to conclude coherently that the results are aligned with the results of the groups of tests 6 and 7, respectively. The differences observed in the progressive behavior of the cycle length and the order quantities are motivated by the impact of the integer multipliers. For instance, one can observe that the order quantities of test cases 11, 12 and 13 of group of tests 9 do not decrease systematically, from test case to test case, due to the impact of the integer multipliers. Actually, the groups of tests 9 and 10, based on base case IV, were purposely performed here in order to "stimulate" the occurrence of optimal integer multipliers greater than one.

Therefore, two sets of tests were performed with the precise model. The first set of tests (based on base cases I and II of Table 1) was used to analyze the performance of our model against the results obtained by Afonso et al. (2022) with the approximate model. The results obtained allowed us to conclude that both models are aligned. The second set of tests (based on base cases III and IV of Table 1) was used to provide some more results exploring the behavior of the model with the optimal integer multipliers not necessarily being equal to one. The sensitivity analyses performed produced similar and expectable results.

In summary, the tests executed with the precise model let us conclude that the increase of the obsolescence risk decreases the period between successive replenishments, meaning that we have to order smaller quantities to prevent against high inventory levels of items subject to obsolescence. The same behavior occurs with the progressive increase of the discount rate. Also, the increase of the major setup cost increases the period between successive replenishments and, consequently, the order quantities. The same occurs with the increase of the minor setup cost because if the ratio $\frac{\sum setup\ costs}{\sum holding\ costs}$ increases substantially, then the replenishments will probably occur less frequently (Silver et al., 2017). As far as a progressive increase of the holding costs and the unit costs are concerned, the model leads to a progressive decrease of the period between successive replenishments and the ordering quantities as well. Progressively increasing the demand implies a decreasing of the period between successive replenishments and higher order quantities.

5. Conclusions

This article developed a precise model of the JRP assuming items subject to obsolescence along an infinite planning horizon. The developed model extended a single-item obsolescence model by assuming a more complex context of multiple items subject to obsolescence which optimizes the total expected discounted cost criterion under the JRP context. Some assumptions are inherited from the classical JRP, such as constant demand, no shortage allowed, no quantity discounts, linear holding cost, and instantaneous delivery. Costs are discounted through an appropriate discount rate and the risk of obsolescence is incorporated by assuming uncertain lifetimes of the multiple items, which follow negative exponential distributions. The model needs a recursive procedure to be evaluated.

Sensitivity analysis has been performed on the model parameters in order to analyze their impact on the ordering policies. In this way, the model was implemented and tested against the results previously obtained with an approximate model. The results of the precise and the approximate models are aligned, meaning that an increase of the risk of obsolescence, through the increase of the obsolescence rates of the items, causes a decrease of the period time between successive replenishments as well as the ordering quantities. The same impact occurs when we face a progressive increase of the discount rate.

The precise model has several cost terms which increase with the number of items. Thus, the optimization process may have a high computational burden and, consequently, may be time consuming. Strategies to overcome these potential shortcomings can be investigated, for instance, by decreasing the number of terms through the removal of terms of the objective function that are expected to have values very close to zero, or by developing heuristics which provide approximations to the optimal solution.

Acknowledgement

This work has been funded by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., Project UIDB/05037/2020.

References

Afonso, R., Godinho, P., & Costa, J. P. (2022). Discounted costs and obsolescence with the joint replenishment problem. *International Journal of Industrial Engineering Computations*, *13*(1), 51–66.

Arcelus, F. J., Pakkala, T. P. M., & Srinivasan, G. (2002). A myopic policy for the gradual obsolescence problem with price-dependent demand. *Computers & Operations Research*, *29*(9), 1115–1127.

Axsäter, S. (2015). Coordinated ordering. In C. C. Price, J. Zhu, & F. S. Hillier (Eds.), *Inventory Control* (3rd ed., Vol. 225, pp. 121–145). Springer International Publishing Switzerland.

Bakker, M., Riezebos, J., & Teunter, R. H. (2012). Review of inventory systems with deterioration since 2001. *European Journal of Operational Research*, *221*(2), 275–284.

Bartels, B., Ermel, U., Sandborn, P., & Pecht, M. G. (2012). *Strategies to the prediction, mitigation and management of product obsolescence* (A. P. Sage (ed.)). John Wiley & Sons, Inc.

Bastos, L., Mendes, M., Nunes, D., Melo, A., & Carneiro, M. (2017). A systematic literature review on the joint replenishment problem solutions: 2006-2015. *Production*, *27*(e20162229), 1–11.

Berk, E., & Gurler, U. (2017). Inventory theory. In J. Sengupta, Raghu; Gupta, Aparna; Dutta (Ed.), *Decision Sciences Theory and Practice* (pp. 349–411). CRC Press - Taylor & Francis Group.

Brown, G. W., Lu, J. Y., & Wolfson, R. J. (1964). Dynamic modeling of inventories subject to obsolescence. *Management Science*, *11*(1), 51–63.

Covert, R. P., & Philip, G. C. (1973). An EOQ model for items with Weibull distribution deterioration. *A I I E Transactions*, *5*(4), 323–326.

Dagpunar, J. S. (1982). Formulation of a multi item single supplier inventory problem. *Journal of the Operational Research Society*, *33*(3), 285–286.

Dohi, T., & Osaki, S. (1995). Optimal inventory policies under product obsolescent circumstance. *Computers & Mathematics with Applications*, *29*(3), 23–30.

Goyal, S. K., & Giri, B. C. (2001). Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research*, *134*(1), 1–16.

Harris, F. W. (1913). How many parts to make at once. *Factory, The Magazine of Management*, *10*(2), 135–136, 152.

Holmbom, M., & Segerstedt, A. (2014). Economic order quantities in production: from Harris to economic lot scheduling problems. *International Journal of Production Economics*, *155*, 82–90.

Joglekar, P., & Lee, P. (1993). An exact formulation of inventory costs and optimal lot size in face of sudden obsolescence. *Operations Research Letters*, *14*(5), 283–290.

Khanlarzade, N., Yegane, B. Y., Kamalabadi, I. N., & Farughi, H. (2014). Inventory control with deteriorating items: a state-of-the-art literature review. *International Journal of Industrial Engineering Computations*, *5*(2), 179–198.

Khouja, M., & Goyal, S. K. (2008). A review of the joint replenishment problem literature: 1989-2005. *European Journal of Operational Research*, *186*(1), 1–16.

Masters, J. M. (1991). A Note on the effect of sudden obsolescence on the optimal lot size. *Decision Sciences*, *22*(5), 1180–1186.

Nahmias, S. (2011). Perishable inventory systems. In *International Series in Operations Research & Management Science, Vol. 160* (1st ed.). Springer US.

Pahl, J., & Voβ, S. (2014). Integrating deterioration and lifetime constraints in production and supply chain planning: a survey. *European Journal of Operational Research*, *238*(3), 654–674.

Sandborn, P. (2013). Design for obsolescence risk management. *Procedia CIRP*, *11*, 15–22.

Silver, E. A., Pyk, D. F., & Thomas, D. J. (2017). Coordinated replenishments at a single stocking point. In *Inventory and Production Management in Supply Chains* (4th ed., Vol. 1, pp. 437–485). CRC Press - Taylor & Francis Group.

Song, J.-S., & Zipkin, P. H. (1996). Managing inventory with the prospect of obsolescence. *Operations Research*, *44*(1), 215–222.

Taft, E. W. (1918). The most economical production lot. *The Iron Age*, *101*(18), 1410–1412.

van Delft, C., & Vial, J. P. (1996). Discounted costs, obsolescence and planned stockouts with the EOQ formula. *International Journal of Production Economics*, *44*(3), 255–265.

Wee, H.-M. (1997). A replenishment policy for items with a price-dependent demand and a varying rate of deterioration. *Production Planning & Control*, *8*(5), 494–499.

Appendix A

Discounted cost contribution to the value of $V(S)$

Suppose C is the numerator corresponding to a cost component over the denominator $\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)$, i.e., $\frac{C}{\left(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T}\right)}$. Then,

$$\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} = \frac{C}{(1-e^{-(\delta+\theta_1+\cdots+\theta_N)T})}$$

Proof:

$$\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} = e^{-(\delta+\theta_1+\cdots+\theta_N)T}\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)(u-1)T} \Leftrightarrow$$

$$e^{(\delta+\theta_1+\cdots+\theta_N)T}\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} = \sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)(u-1)T} \Leftrightarrow$$

$$e^{(\delta+\theta_1+\cdots+\theta_N)T}\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} - \sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} = \sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)(u-1)T} - \sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} \Leftrightarrow$$

$$\left(e^{(\delta+\theta_1+\cdots+\theta_N)T}-1\right)\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} = C\left(\sum_{u=-1}^{\infty} e^{-(\delta+\theta_1+\cdots+\theta_N)uT} - \sum_{u=0}^{\infty} e^{-(\delta+\theta_1+\cdots+\theta_N)uT}\right) \Leftrightarrow$$

$$\left(e^{(\delta+\theta_1+\cdots+\theta_N)T}-1\right)\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} = Ce^{(\delta+\theta_1+\cdots+\theta_N)T} \Leftrightarrow$$

$$\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} = \frac{Ce^{(\delta+\theta_1+\cdots+\theta_N)T}}{(e^{(\delta+\theta_1+\cdots+\theta_N)T}-1)} \Leftrightarrow$$

$$\sum_{u=0}^{\infty} Ce^{-(\delta+\theta_1+\cdots+\theta_N)uT} = \frac{C}{(1-e^{-(\delta+\theta_1+\ldots+\theta_N)T})}$$

as we wanted to prove.