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# A slightly compressible hyperelastic material model implementation in ABAQUS

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**A B S T R A C T**

ABAQUS/Standard is a powerful finite element program designed for general use in nonlinear problems. The paper touches only one aspect of usage of the software, namely a constitutive modelling of slightly compressible hyperelastic materials. It begins with a discussion of a well-known approach of describing slight compressibility in the context of the stored energy function. Basic equations of continuum mechanics are presented as well. The main part of the work concerns an implementation of one of the presented models. To this end, the UHYPER user subroutine is employed. Analytical formulas for a few simple, homogeneous deformation states are given which allow verifying numerical results. Finally, a couple demonstration examples with nonhomogeneous deformations are presented as an attempt at motivating applications in engineering.



## 1. Introduction

The aim of this paper is to describe a generalization of a certain class of hyperelastic incompressible material models (Jemioło, 2016). The generalization is provided by extending basic model to capture slight compressibility of a material. These models are used to predict the response of elastomers (or more general co called rubber-like materials) under static or dynamic loading conditions (Ogden, 1984). Rubber-like materials are used in automotive parts such as tires, engine and transmission mounts, center bearing supports and exhaust rubber parts. Nowadays, the design of these highly technical parts necessitates the use of simulation tools such as finite element method (FEM) software (Zienkiewicz, Taylor, & Fox, 2014). In this context, an appropriate constitutive model is an essential prerequisite for accurate numerical predictions (James, Green, & Simpson, 1975).

In the case of rubber-like materials, the volumetric compressibility modulus is couple orders of magnitude larger than the shear modulus $\mathrm{K}_0 >> \mu_0$ (Chadwick, 1974). Therefore, when interpreting typical experimental results of uniaxial and biaxial stretching and simple shear, universal relationships for an incompressible material model are used (Jemioło & Franus, 2019). In order to take into account the compressibility of a material, the stored energy function (SEF) is assumed to be sum of two functions, i.e. the function of the isochoric deformation $\bar{W}$ and the volumetric one $W_{vol}$. Such approach leads to a

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clear physical interpretation when describing material behaviour and it is suitable for defining slightly compressible materials.

A significant number of papers proposed constitutive equations for rubber-like materials (Levinson & Burgess, 1971). We focus on a class of material models (MCMV) that was proposed in the authors' work (Jemioło, 2019). In particular, we present an implementation for one of them, namely MCIZ model, in the ABAQUS/Standard using the UHYPER user subroutine (Dassault Systèmes, 2015). In order to verify the implementation, basic numerical tests are performed. Obtained results are compared to values following from analytical formulas. Moreover, an application of the MCIZ model is shown by solution of exemplary boundary value problems.

A printout of the UHYPER subroutine of the MCIZ material model, written in Fortran language, can be found in the Appendix.

## 2. Constitutive relationships of isotropic hyperelasticity

### *2.1 Incompressible materials*

In the case of an incompressible material, the stored energy function is not a potential of elasticity, because the volumetric part of the Cauchy stress tensor is not determined (Ciarlet, 1988). The material may be subjected only to isochoric deformations, i.e. deformations without changing the volume of the body. Each deformation of the incompressible body is defined by the condition

$$J - 1 = 0, \tag{1}$$

where $J = \det \mathbf{F}$ is determinant of so-called the deformation gradient $\mathbf{F}$. Consequently, the elastic potential must incorporate the constraint or Eq. (1) with a Lagrange multiplier, which should be interpreted as hydrostatic pressure. The tensor $\mathbf{F}$ is decomposed into the isochoric and volumetric part as follows:

$$\mathbf{F} = J^{1/3}\overline{\mathbf{F}}, \quad \det \overline{\mathbf{F}} = 1. \tag{2}$$

Further, we introduce

$$\overline{\mathbf{B}} = \overline{\mathbf{F}}\overline{\mathbf{F}}^T, \quad \overline{\mathbf{C}} = \overline{\mathbf{F}}^T\overline{\mathbf{F}}, \tag{3}$$

where $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$ denote the left and the right Cauchy-Green deformation tensors of isochoric deformations (Holzapfel, 2010).

In the case of incompressible isotropic materials, the stored energy function (SEF) $W$ is a function of only two independent invariants of isochoric deformations such that

$$W = W\left(\overline{\mathbf{C}}\right) = W\left(\overline{\mathbf{B}}\right) = \overline{W}\left(\overline{I}_1, \overline{I}_2\right), \tag{4}$$

where

$$\overline{I}_1 = \operatorname{tr}\overline{\mathbf{B}} = \operatorname{tr}\overline{\mathbf{C}}, \quad \overline{I}_2 = \operatorname{tr}\overline{\mathbf{B}}^{-1} = \operatorname{tr}\overline{\mathbf{C}}^{-1}. \tag{5}$$

Using equations (1) and (4) we obtain the following constitutive relationship of hyperelasticity in spatial description:

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\frac{\partial \overline{W}\left(\overline{I}_1, \overline{I}_2\right)}{\partial \overline{I}_1}\overline{\mathbf{B}} - 2\frac{\partial \overline{W}\left(\overline{I}_1, \overline{I}_2\right)}{\partial \overline{I}_2}\overline{\mathbf{B}}^{-1} \equiv -p\mathbf{I} + \overline{\beta}_1\overline{\mathbf{B}} + \overline{\beta}_{-1}\overline{\mathbf{B}}^{-1}. \tag{6}$$

Basic invariants $\bar{I}_1$ and $\bar{I}_2$, due to the constraints of incompressibility (1), can be considered as functions of only two independent eigenvalues $\bar{\lambda}_1$ and $\bar{\lambda}_2$ of the stretch tensors $\bar{\mathbf{U}} = \sqrt{\bar{\mathbf{C}}}$ and $\bar{\mathbf{V}} = \sqrt{\bar{\mathbf{B}}}$

$$\bar{I}_1 = \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \left(\bar{\lambda}_1\bar{\lambda}_2\right)^{-2}, \qquad \bar{I}_2 = \bar{\lambda}_1^{-2} + \bar{\lambda}_2^{-2} + \left(\bar{\lambda}_1\bar{\lambda}_2\right)^{2}. \tag{7}$$

The constrain of incompressibility implies that $\bar{\lambda}_3 = \left(\bar{\lambda}_1\bar{\lambda}_2\right)^{-1}$. Principal stretches $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are independent but they are unordered eigenvalues of stretch tensors. It is easy to check that the function $\bar{I}_1$ is a convex function with respect to $\bar{\lambda}_1$ and $\bar{\lambda}_2$ while the function $\bar{I}_2$ is not a convex function with respect to $\bar{\lambda}_1$ and $\bar{\lambda}_2$, for sufficiently large deformations (Jemioło, 2016).

*2.2 Slightly compressible materials*

Models of isotropic slightly compressible materials are a generalisation of the models describing incompressible materials in which there is no coupling between the stored energy function of isochoric $\bar{W}$ and volumetric $W_{vol}$ deformations (Jemioło, 2019). To the function $\bar{W}$ of incompressible material models, dependent on two invariants of the isochoric deformation $\bar{I}_1$ and $\bar{I}_2$, a sufficiently regular function $W_{vol}(J)$ is added, dependent on the invariant describing volumetric deformation $J = \det \mathbf{F}$. Consequently, the SEF takes the form

$$W = \bar{W}\left(\bar{I}_1, \bar{I}_2\right) + W_{vol}(J). \tag{8}$$

In the case of hyperelastic isotropic materials, the constitutive relationships in the Eulerian description (for the deformed configuration) can be obtained from the following relationships:

$$\boldsymbol{\sigma} = \frac{2}{J}\frac{\partial W}{\partial \mathbf{B}}\bigg|_{\mathbf{B}=\mathbf{B}^T} \mathbf{B} = \frac{2}{J}\mathbf{B}\frac{\partial W}{\partial \mathbf{B}}\bigg|_{\mathbf{B}=\mathbf{B}^T}, \tag{9}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{B}$ is the left deformation tensor ($\mathbf{B} = \mathbf{F}\mathbf{F}^T$, the symbol „T" denotes the transposition of a tensor). Due to the form of SEF (8), the constitutive relationship of a slightly compressible material is divided into two independent parts:

*the deviatoric part*:

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{I} = \frac{2}{J}\left(\bar{\alpha}_1\bar{\mathbf{B}}_D - \bar{\alpha}_2\bar{\mathbf{B}}_D^{-1}\right), \tag{10}$$

where

$$\bar{\alpha}_1 = \frac{\partial \bar{W}}{\partial \bar{I}_1}, \quad \bar{\alpha}_2 = \frac{\partial \bar{W}}{\partial \bar{I}_2}, \tag{11}$$

*and the volumetric part:*

$$\mathrm{tr}\,\boldsymbol{\sigma} = 3\frac{\partial W_{vol}}{\partial J}. \tag{12}$$

We remind that the modified invariants $\bar{I}_1 = \mathrm{tr}\bar{\mathbf{B}}$, $\bar{I}_2 = \mathrm{tr}\bar{\mathbf{B}}^{-1}$ of $\bar{\mathbf{B}} = J^{-2/3}\mathbf{B}$ are the same for deformation $\bar{\mathbf{C}} = J^{-2/3}\mathbf{C}$, where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green deformation tensor. These can be explicitly written as

$$\bar{I}_1 = J^{-\frac{2}{3}} I_1, \quad \bar{I}_2 = J^{-\frac{4}{3}} I_2, \tag{13}$$

where

$$I_1 = \operatorname{tr} \mathbf{B} = \operatorname{tr} \mathbf{C}, \quad I_2 = \frac{1}{2}\left(I_1^2 - \operatorname{tr} \mathbf{B}^2\right), \quad J = \sqrt{\det \mathbf{B}} = \sqrt{\det \mathbf{C}}. \tag{14}$$

Tensor $\bar{\mathbf{B}}_D$ stands for deviatoric parts of the modified left Cauchy-Green tensor:

$$\bar{\mathbf{B}}_D = \bar{\mathbf{B}} - \frac{1}{3}\bar{I}_1 \mathbf{I}, \quad \bar{\mathbf{B}}_D^{-1} = \bar{\mathbf{B}}^{-1} - \frac{1}{3}\bar{I}_2 \mathbf{I}. \tag{15}$$

Constitutive relations in the initial configuration may be obtained using (10)–(12) and the following relations:

$$\mathbf{S} = J\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad \mathbf{T} = \mathbf{F}^{-1}\mathbf{S}, \tag{16}$$

where **S** and **T** are the first and the second Piola-Kirchhoff stress tensors, respectively.

## 3. Slightly compressible material models

### *3.1. Polynomial model in ABAQUS*

The ABAQUS's library (Dassault Systèmes, 2015) contains the hyperelastic material model according to the modified Rivlin polynomial (Rivlin, 1948). It is a generalization of the SEF of the incompressible model in the following form:

$$W = \bar{W}\left(\bar{I}_1, \bar{I}_2\right) + W_{vol}(J) = \sum_{k+l=1}^{N} C_{kl}\left(\bar{I}_1 - 3\right)^k \left(\bar{I}_2 - 3\right)^l + \sum_{k=1}^{N} \frac{1}{D_k}(J-1)^{2k}, \tag{17}$$

where the parameters $D_k$ determine the compressibility of the material (for $D_k \to 0$ the model of incompressible material is restored). It is worth to mention that the initial shear and bulk modules for (17) are given by

$$\mu_0 = 2\left(C_{10} + C_{01}\right), \quad \mathrm{K}_0 = \frac{2}{D_1} \tag{18}$$

The most popular and easiest models of rubber-like materials Mooney-Rivlin (MR) (Mooney, 1940) and neo-Hookean (NH) are special cases of the function (17):

$$W = \bar{W}\left(\bar{I}_1, \bar{I}_2\right) + W_{vol}(J) = C_{10}\left(\bar{I}_1 - 3\right) + C_{01}\left(\bar{I}_2 - 3\right) + \frac{1}{D_1}(J-1)^2 \tag{19}$$

where Eq. (18) holds true. In the case of NH model, we have $C_{01} = 0$.

In the literature it is believed that Rivlin's proposal has little practical significance, because it contains too many material parameters and in general does not lead to polyconvex SEF (Ball, 1977). Nevertheless, the Rivlin form is a starting point for the search for simplified models, also in polynomial form, such as model MV and its special cases discussed below.

Formally, the generalised Yeoh model (Yeoh, 1990)

$$W = \overline{W}\left(\overline{I}_1\right) + W_I\left(J\right) = \sum_{k=1}^{N} C_{k0}\left(\overline{I}_1 - 3\right)^k + \sum_{k=1}^{N} \frac{1}{D_k}\left(J-1\right)^{2k} \tag{20}$$

is a special case of (17). Due to the specific form of the SEF, it significantly facilitates the identification of material parameters and is recommended by the authors of ABAQUS (Dassault Systèmes, 2015). For conclusions on the limitations of Rivlin and Yeoh models we refer the reader to (Jemioło & Franus, 2019).

*3.2. Model MCMV and its special cases*

We consider the generalizations of the MV model (Jemioło, 2016) with the following SEF:

$$\overline{W}\left(\overline{I}_1, \overline{I}_2\right) = \frac{1}{2}\left[a_1\left(\overline{I}_1 - 3\right) + \frac{1}{2}a_2\left(\overline{I}_1^2 - 9\right) + \frac{1}{3}a_3\left(\overline{I}_1^3 - 27\right) + a_4\left(\overline{I}_2 - 3\right) + a_5\left(\overline{I}_1\overline{I}_2 - 9\right)\right], \tag{21}$$

where $a_k\ (k = 1,\ldots,5)$ are parameters of the model. The deviatoric part of stored energy function of the MCMV model can be written in equivalent form

$$\overline{W} = U\left(\overline{I}_1, \overline{I}_2\right) = C_{10}\left(\overline{I}_1 - 3\right) + C_{20}\left(\overline{I}_1 - 3\right)^2 + C_{30}\left(\overline{I}_1 - 3\right)^3 + C_{01}\left(\overline{I}_2 - 3\right) + C_{11}\left(\overline{I}_1 - 3\right)\left(\overline{I}_2 - 3\right), \tag{22}$$

where

$$C_{10} = \frac{1}{2}\left(a_1 + 3a_2 + 9a_3 + 3a_5\right),\ C_{20} = \frac{1}{4}\left(a_2 + 6a_3\right),\quad C_{30} = \frac{1}{6}a_3,\ C_{01} = \frac{1}{2}\left(a_4 + 3a_5\right),\quad C_{11} = \frac{1}{2}a_5, \tag{23}$$

or inversely

$$\begin{aligned} &a_1 = 2\left(C_{10} - 3C_{11} - 6C_{20} + 27C_{30}\right),\quad a_2 = 4\left(C_{20} - 9C_{30}\right),\quad a_3 = 6C_{30},\\ &a_4 = 2\left(C_{01} - 3C_{11}\right),\quad a_5 = 2C_{11}. \end{aligned} \tag{24}$$

The functions in constitutive relation (10) are of the form:

$$\begin{aligned} &\overline{\alpha}_1 = \frac{1}{2}\overline{\beta}_1,\quad \overline{\alpha}_2 = -\frac{1}{2}\overline{\beta}_2,\quad \overline{\beta}_1 = 2\frac{\partial U}{\partial \overline{I}_1} = a_1 + a_2\overline{I}_1 + a_3\overline{I}_1^2 + a_5\overline{I}_2,\\ &\overline{\beta}_{-1} = -2\frac{\partial U}{\partial \overline{I}_2} = -a_4 - a_5\overline{I}_1. \end{aligned} \tag{25}$$

Moreover, the initial shear modulus is given by

$$\mu_0 = a_1 + 3a_2 + 9a_3 + a_4 + 6a_5 = 2\left(C_{10} + C_{01}\right) \tag{26}$$

From relation (21) other well-known stored energy functions of incompressible hyperelastic material models may be obtained. By neglecting the parameter with the coupling of the first and second invariant of isochoric deformation we have the SEF equivalent to the Biderman model (MB) (Biderman, 1958). If we additionally omit the second invariant an equivalent form of the Yeoh model (MY) is recovered. On the order hand, taking $a_2 = a_3 = 0$ yields the Mooney-Rivlin model (MR). The simplest SEF of incompressible material is the form of neo-Hookean (NH) model, where in (21) there is only one non-zero elasticity constant of the interpretation of the initial shear modulus (identical to Hooke's law of linear elasticity), i.e. $a_1 = \mu_0$.

Setting $a_3 = 0$, $a_5 = 0$ Ishihara-Zahorski model is recovered (MIZ) (Isihara, Hashitsume, & Tatibana, 1951), (Zahorski, 1959). The stored energy function can be expressed in the form

$$\overline{W}\left(\overline{I}_1,\overline{I}_2\right)=\frac{1}{2}\mu_0\left[f\left(\overline{I}_1-3\right)+\left(1-f\right)\left(\overline{I}_2-3\right)+\frac{1}{2}\overline{c}\left(\overline{I}_1-3\right)^2\right] \tag{27}$$

with the initial shear modulus $\mu_0>0$ and two material parameters such that $f\in\left(0,1\right)$, $\overline{c}>0$.
We emphasize that functions (21) and (27) are respectively consistent third and second order approximation of existing stored energy function in term of $\left\|\overline{\mathbf{C}}\right\|$, cf. (Jemioło & Franus, 2019 and see Fig. 1).

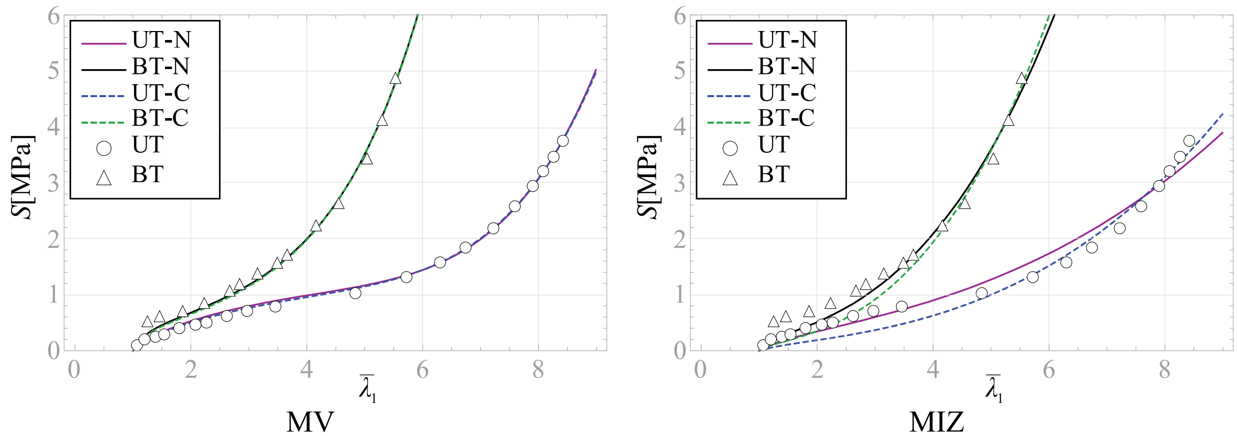


**Fig. 1** The MV and MIZ models. Nominal stress – principal stretch for uniaxial (UT) and biaxial (BT) tensile test. Comparison of approximation based on nominal stress (N) and Cauchy (C) respectively with experimental results (Alexander, 1968).

Examples of parameter values for discussed models, namely MV and MIZ, are summarized in Table 1. For details we refer the reader to (Franus & Jemioło, 2019).

**Table 1.** Parameters of MV and MIZ for neoprene (Alexander, 1968).

| Model | MV | MIZ |
|---|---|---|
| $a_1$ [MPa] | 3.152 $10^{-1}$ | 1.710 $10^{-1}$ |
| $a_2$ [MPa] | –6.469 $10^{-3}$ | 3.218 $10^{-3}$ |
| $a_3$ [MPa] | 1.173 $10^{-4}$ | - |
| $a_4$ [MPa] | 1.899 $10^{-2}$ | 1.547 $10^{-2}$ |
| $a_5$ [MPa] | –3.011 $10^{-5}$ | - |
| Variance | 7.470 $10^{-3}$ | 3.916 $10^{-2}$ |

### *3.3 Volumetric stored energy function*

The part of SEF describing volumetric changes should meet certain mathematical requirements. Firstly, we assume that $W_{vol}\left(J\right)$ is differentiable with respect to *J* and $W_{vol}\left(1\right)=0$. Based on assumption of existence of natural state it follows that

$$\left.\frac{\partial W_{vol}\left(J\right)}{\partial J}\right|_{J=1}=0. \tag{28}$$

Secondly, $W_{vol}\left(J\right)$ should be a convex function, i.e.

$$\frac{\partial^2 W_{vol}(J)}{\partial J^2} \geq 0. \tag{29}$$

To ensure proper growth, the conditions should hold

$$\begin{aligned} &W_{vol}(J \to 0) = +\infty, \quad \left.\frac{\partial W_{vol}(J)}{\partial J}\right|_{J\to 0} = -\infty, \\ &W_{vol}(J \to +\infty) = +\infty, \quad \left.\frac{\partial W_{vol}(J)}{\partial J}\right|_{J\to +\infty} = +\infty. \end{aligned} \tag{30}$$

In this case, the initial bulk modulus is given by

$$\mathrm{K}_0 = \frac{\partial^2 W_{vol}}{\partial J^2}\Big|_{J=1}. \tag{31}$$

**Table 2.** Forms of volumetric part of stored energy function

| No. | $W_{vol}(J)$ |
|---|---|
| **1** | $\mathrm{K}_0\left[\frac{1}{4}(J^2-1)-\frac{1}{2}\ln J\right]$ |
| **2** | $\frac{\mathrm{K}_0}{4}\left[(J-1)^2+(\ln J)^2\right]$ |
| **3** | $\frac{\mathrm{K}_0}{n^2}\left[n\ln J+J^{-n}-1\right], \quad n<-1$ |
| **4** | $\mathrm{K}_0(a+b)\left(\frac{J^{a+1}}{a+1}+\frac{J^{1-b}}{b-1}\right)-\frac{\mathrm{K}_0}{(a+1)(b-1)}, \quad a>0, \quad b>1$ |
| **5** | $\frac{\mathrm{K}_0}{2}(J-1)\ln J$ |
| **6** | $\frac{\mathrm{K}_0}{2}\left(e^{J-1}-\ln J-1\right)$ |

We emphasize that the function (17) implemented in ABAQUS/Standard do not meet presented conditions. Examples of known in literature $W_{vol}(J)$ (Doll & Schweizerhof, 1999) that meet the above-mentioned mathematical conditions are presented in Table 2 and plotted in Fig. 2. The functions 1,2 and 5,6 are scaled by the initial bulk modulus (identical to the one from the linear elastic theory), i.e. only one experimental test is needed to determine them, which can be performed in the low strain range. Other functions also include additional material parameters.

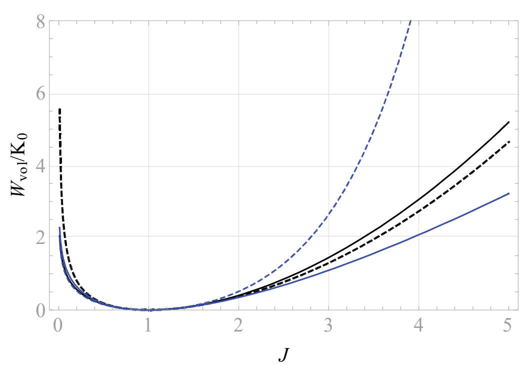


**Fig. 2** Plots of functions shown in Table 2: 1 – black line, 2 – black dashed line, 5 –blue line, 6 –blue dashed line

## 4. Implementation of the model

### *4.1 Uhyper procedure description*

In ABAQUS/Standard and the UHYPER procedure (Jemioło & Franus, 2018), the multiplicative decomposition of the deformation gradient tensor into the volumetric and isochoric parts is used, i.e

$$\mathbf{F} = \left(\sqrt[3]{J}\mathbf{I}\right)\overline{\mathbf{F}}, \tag{32}$$

where $J = \det \mathbf{F}$. We remind definitions of deformation tensors and their invariants:

$$\begin{aligned} &\overline{\mathbf{B}} = \overline{\mathbf{F}}\overline{\mathbf{F}}^T, \quad \overline{\mathbf{C}} = \overline{\mathbf{F}}^T\overline{\mathbf{F}}, \\ &\overline{I}_1 = \mathrm{tr}\overline{\mathbf{B}} = \mathrm{tr}\overline{\mathbf{C}}, \quad \overline{I}_2 = \mathrm{tr}\overline{\mathbf{B}}^{-1} = \mathrm{tr}\overline{\mathbf{C}}^{-1}. \end{aligned} \tag{33}$$

As the considered generalized MIZ model, namely the MCIZ, is not available in the standard ABAQUS's library, the UHYPER procedure is employed to define such stored energy function, see Appendix. It requires derivatives of the SEF with respect to the modified invariants $\overline{I}_1, \overline{I}_2$ and *J*. In the case of the function (27) these are given (non-zero derivatives) as follows:

-first order derivatives

$$\begin{aligned} &\frac{\partial W}{\partial \overline{I}_1} = \frac{\partial \overline{W}}{\partial \overline{I}_1} = \frac{1}{2}\mu_0\left[f + \overline{c}\left(\overline{I}_1 - 3\right)\right], \\ &\frac{\partial W}{\partial \overline{I}_2} = \frac{\partial \overline{W}}{\partial \overline{I}_2} = \frac{1}{2}\mu_0\left(1 - f\right), \\ &\frac{\partial W}{\partial J} = \frac{\partial W_{vol}}{\partial J} = \mathrm{K}_0\frac{J^2 - 1}{2J}, \end{aligned} \tag{34}$$

-second order derivatives

$$\begin{aligned} &\frac{\partial^2 W}{\partial \overline{I}_1^2} = \frac{\partial^2 \overline{W}}{\partial \overline{I}_1^2} = \frac{1}{2}\mu_0\overline{c}, \\ &\frac{\partial^2 W}{\partial J^2} = \frac{\partial^2 W_{vol}}{\partial J^2} = \mathrm{K}_0\frac{J^2 + 1}{2J^2}, \end{aligned} \tag{35}$$

-third order derivatives

$$\frac{\partial^3 W}{\partial J^3} = \frac{\partial^3 W_{vol}}{\partial J^3} = -\mathrm{K}_0\frac{1}{J^3}. \tag{36}$$

It is important to recall that knowing the first order derivatives in relation to considered invariants we know the constitutive relation of a material model in the current (deformed) configuration of the body:

$$\boldsymbol{\sigma} = \overline{\alpha}_0\mathbf{I} + \frac{2}{J}\left(\overline{\alpha}_1\left(\overline{\mathbf{B}} - \frac{1}{3}\overline{I}_1\mathbf{I}\right) - \overline{\alpha}_2\left(\overline{\mathbf{B}}^{-1} - \frac{1}{3}\overline{I}_2\mathbf{I}\right)\right) \tag{37}$$

with

$$\bar{\alpha}_0 = \frac{\partial W_{vol}}{\partial J}, \quad \bar{\alpha}_1 = \frac{\partial \bar{W}}{\partial \bar{I}_1}, \quad \bar{\alpha}_2 = \frac{\partial \bar{W}}{\partial \bar{I}_2} \tag{38}$$

In the case of incompressible materials, the above relations are simplified (Suchocki & Jemioło, 2019). The constitutive relation of such materials takes the form

$$\boldsymbol{\sigma} = -p\mathbf{I} + \bar{\beta}_1 \bar{\mathbf{B}} + \bar{\beta}_{-1} \bar{\mathbf{B}}^{-1}, \tag{39}$$

where $p$ is an unknown hydrostatic pressure and we have

$$\bar{\beta}_1 = 2\frac{\partial \bar{W}}{\partial \bar{I}_1}, \quad \bar{\beta}_{-1} = -2\frac{\partial \bar{W}}{\partial \bar{I}_2}. \tag{40}$$

Hence, an implementation of incompressible material model via UHYPER requires setting the derivatives of SEF with respect to $J$ to zero.

It is worth to mention that in the case of large values of the bulk modulus $\mathrm{K}_0$ in comparison to a value of the initial modulus $\mu_0$, it is recommended in ABAQUS/Standard to use finite elements with a hybrid formulation, see (Brezzi & Fortin, 1991) and (Bonet et al., 2016).

### *4.2 Basic Numerical Tests*

The implementation of the MCIZ model is verified in simple tests with homogeneous deformations, i.e. uniaxial and biaxial states, using CPE4 element (plane strain). A three-dimensional element could also be used. For such homogeneous states, stress tensor's components are easily derived from the constitutive equation, see (Suchocki, 2017). Therefore, the accuracy of numerical results may be straightforward verified.

In the case of biaxial deformation state, the deformation gradient takes the form

$$\mathbf{F} = \lambda_1 \mathbf{b}_1 \otimes \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 \otimes \mathbf{b}_2 + \mathbf{b}_3 \otimes \mathbf{b}_3, \tag{41}$$

with $\det \mathbf{F} = J = \lambda_1 \lambda_2$. In particular, we consider uniaxial deformation state, i.e. $\lambda_2 = 1$, and uniform biaxial state $\lambda_1 = \lambda_2$. Combining equation (41) with (37) we obtain explicit formulas for components of the Cauchy stress tensor:

a) uniaxial deformation state:

$$\begin{aligned} \sigma_{11} &= \frac{\lambda_1^2 - 1}{6\lambda_1^{7/3}}\left[4\mu_0\left(1 - f + 2c + \left(f - 3c\right)\lambda_1^{2/3} + c\lambda_1^2\right) + 3\mathrm{K}_0 \lambda_1^{4/3}\right] \\ \sigma_{22} = \sigma_{33} &= \frac{1 - \lambda_1^2}{6\lambda_1^{7/3}}\left[2\mu_0\left(1 - f + 2c + \left(f - 3c\right)\lambda_1^{2/3} + c\lambda_1^2\right) - 3\mathrm{K}_0 \lambda_1^{4/3}\right]. \end{aligned} \tag{42}$$

b) uniform biaxial deformation state:

$$\begin{aligned} \sigma_{11} = \sigma_{22} &= \frac{3\mathrm{K}_0 \lambda_1^2 \left(\lambda_1^4 - 1\right)}{6\lambda_1^4} + \frac{2\mu_0 \left(\lambda_1^2 - 1\right)\left[\lambda_1^2\left(1 - f + 2c\right) + \lambda_1^{4/3}\left(a - 3c\right) + c\right]}{6\lambda_1^{14/3}}, \\ \sigma_{33} &= \frac{3\mathrm{K}_0 \lambda_1^2 \left(\lambda_1^4 - 1\right)}{6\lambda_1^4} - \frac{4\mu_0 \left(\lambda_1^2 - 1\right)\left[\lambda_1^2\left(1 - f + 2c\right) + \lambda_1^{4/3}\left(a - 3c\right) + c\right]}{6\lambda_1^{14/3}}. \end{aligned} \tag{43}$$

The tests are performed setting material parameters as: $f = 0.75, \bar{c} = 0.1, K_0 = 10\mu_0$. In all tested types of deformations, analytical values confirm results based on the finite element procedure within assumed precision of calculations, see Fig. 3 and Fig. 4.

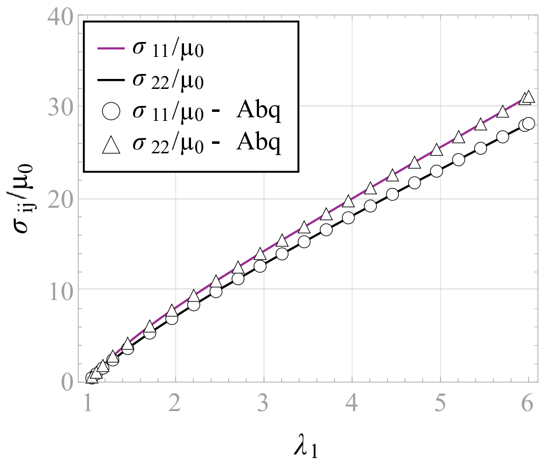


**Fig. 3** Plots of normalized components of the Cauchy stress tensor for uniaxial deformation test

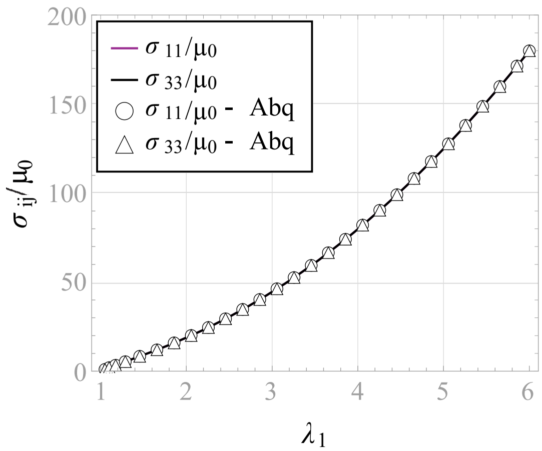


**Fig. 4** Plots of normalized components of the Cauchy stress tensor for biaxial deformation test

In order to illustrate some basic properties of the MCIZ model and its incompressible version (MIZ), we additionally consider uniaxial and biaxial tension tests. For an incompressible material model, relation stress-stretch may be given explicitly. Thus, under the assumption, the deformation gradient for uniaxial compression/tension takes the form

$$\bar{\mathbf{F}} = \bar{\lambda}_1 \mathbf{b}_1 \otimes \mathbf{b}_1 + \bar{\lambda}_1^{-\frac{1}{2}} \left( \mathbf{b}_2 \otimes \mathbf{b}_2 + \mathbf{b}_3 \otimes \mathbf{b}_3 \right). \tag{44}$$

For such deformation there is only one non-zero component of stress tensor in the considered basis $\{\mathbf{b}_i\}$. The value of the nominal stress yields

$$S_1 = \left(1 - \bar{\lambda}_1^{-3}\right)\left(\bar{\lambda}_1 \bar{\beta}_1 - \bar{\beta}_{-1}\right), \tag{45}$$

where for the MIZ model we have

$$\bar{\beta}_1 = \left[ f + \bar{c}\left(2\bar{\lambda}_1^{-1} + \bar{\lambda}_1^2 - 3\right)\right]\mu_0, \quad \bar{\beta}_{-1} = -\left(1 - f\right)\mu_0. \tag{46}$$

In the case of biaxial tension with equal principal stretches $\bar{\lambda}_1 = \bar{\lambda}_2$, the deformation gradient reads

$$\bar{\mathbf{F}} = \bar{\lambda}_1 \left( \mathbf{b}_1 \otimes \mathbf{b}_1 + \mathbf{b}_2 \otimes \mathbf{b}_2 \right) + \bar{\lambda}_1^{-2} \mathbf{b}_3 \otimes \mathbf{b}_3. \tag{47}$$

Hence, the nominal principal stresses are given by

$$S_1 = S_2 = \left(\bar{\lambda}_1 - \bar{\lambda}_1^{-5}\right)\left(\bar{\beta}_1 - \bar{\lambda}_1^2 \bar{\beta}_{-1}\right) \tag{48}$$

with

$$\bar{\beta}_1 = \left[ f + \bar{c}\left(\bar{\lambda}_1^{-4} + 2\bar{\lambda}_1^2 - 3\right)\right]\mu_0, \quad \bar{\beta}_{-1} = -\left(1 - f\right)\mu_0. \tag{49}$$

We remind that for $J = 1$ relation between the nominal stress tensor $\mathbf{S}$ and the Cauchy tensor $\boldsymbol{\sigma}$ is as follows:

$$\mathbf{S} = \boldsymbol{\sigma}\left(\overline{\mathbf{F}}^{-1}\right)^{T} = \boldsymbol{\sigma}\,\mathrm{Cof}\,\overline{\mathbf{F}} \tag{50}$$

Additionally, a comparison of influence of the initial bulk modulus $K_0$ on results of uniaxial and biaxial tension tests are carried out using ABAQUS/Standard for $K_0 = 10\mu_0$ and $K_0 = 1000\mu_0$ with $f = 0.75, \bar{c} = 0.1$. Fig. 5 and Fig. 6 present obtained values of the Cauchy stress tensor's components. In this case it can be noted that in a stretch range of $\lambda_1 < 4$ the results do not differ significantly between $K_0 = 1000\mu_0$ case and the MIZ model for both tests.

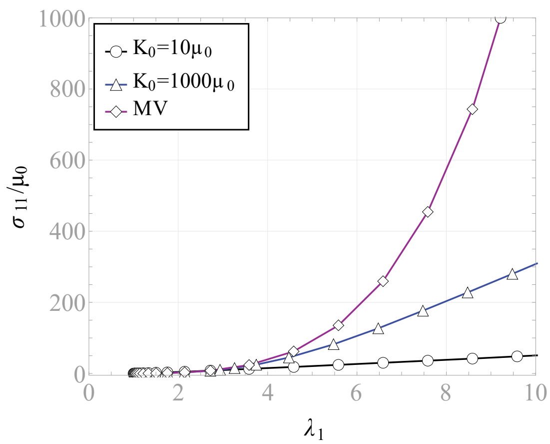


**Fig. 5** Comparison of analytical and numerical solutions for uniaxial tension test

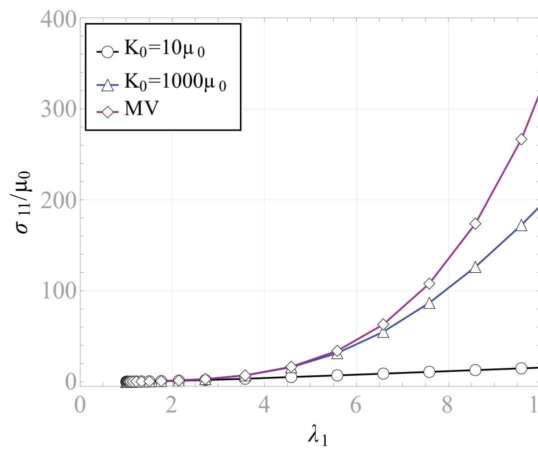


**Fig. 6** Comparison of analytical and numerical solutions for biaxial tension test

In the case of incompressibility (MIZ), analytical values confirm numerically obtained ones within assumed precision of calculations, see Fig. 7.

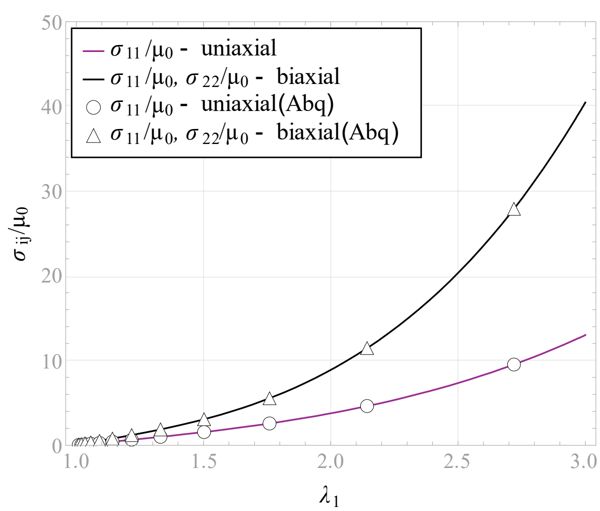


**Fig. 7** Comparison of analytical and numerical solutions for uniaxial and biaxial tests

## 5. Examples

### 5.1 Deformation of a Block

The first boundary value problem illustrating the behavior of the MCMV material model with different compressibility is a block presented in Fig. 8. The FEM model consists of 4000 C3D8H (hybrid formulation) elements. Over the marked face BCDE, prescribed displacement is applied, i.e. $u_2 = -L/2, u_1 = 0$, and zero traction is prescribed over the face ABEF. On the other faces $u_n = 0$ is imposed, where $n$ stands for normal direction to each face. Values of the parameters are taken according to Table 1.

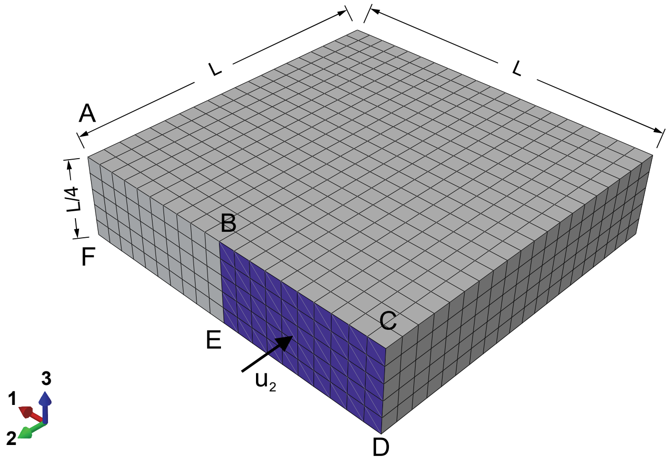


**Fig. 8** Scheme of the considered boundary value problem

Contour plots of displacement $u_2$ in the actual configuration of the body are presented in Fig. 9 and Fig. 10 for the MV and the MCMV model with $K_0 = 10\mu_0$, respectively. Fig.11 presents plots of averaged normal stress $\bar{\sigma}$ as a function of the imposed displacement $u_2$. The values of $\bar{\sigma}$ are obtained by averaging values of the component $\sigma_{22}$ from the Gauss points of the first layer finite elements in the region marked BCDE according to Fig. 8. The results for MV model do not differ significantly from values obtained in the case of the MCMV model with $K_0 = 1000\mu_0$. One can notice considerably lower stiffness when $K_0 = 10\mu_0$ then for the previous cases in the context of considered relationship.

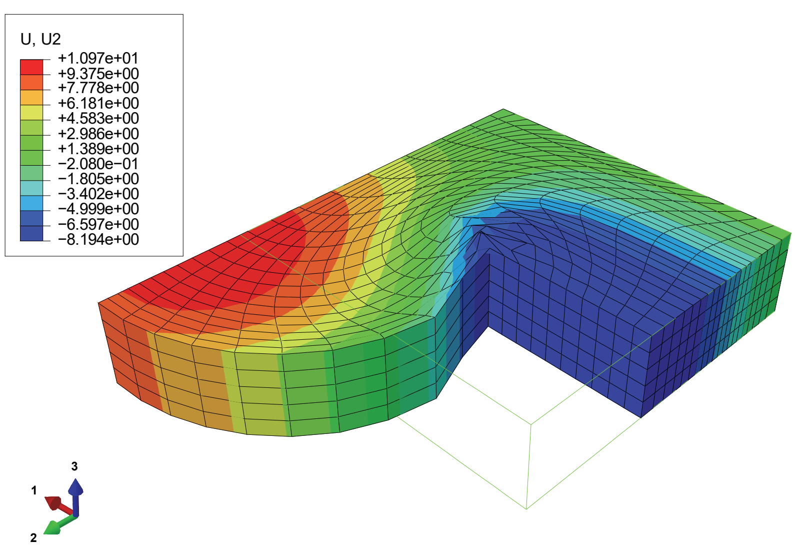


**Fig. 9** Map of $u_2$ in final configuration for MV model

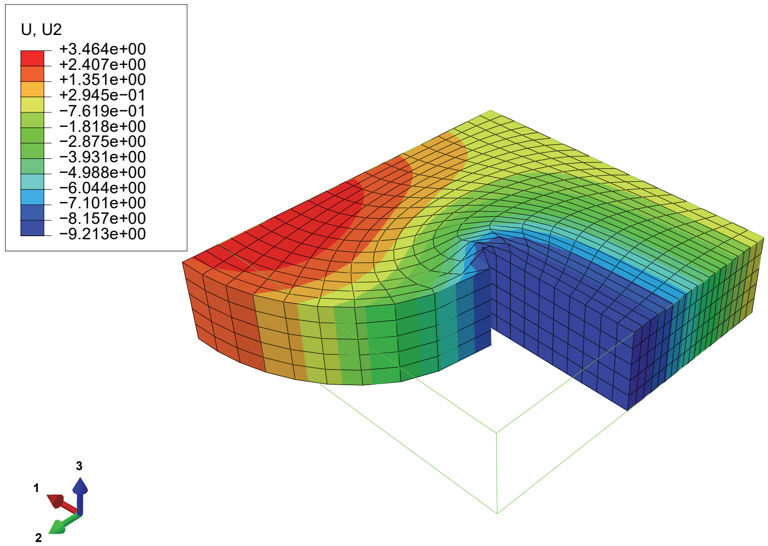


**Fig. 10** Map of $u_2$ in final configuration for $K_0 = 10\mu_0$

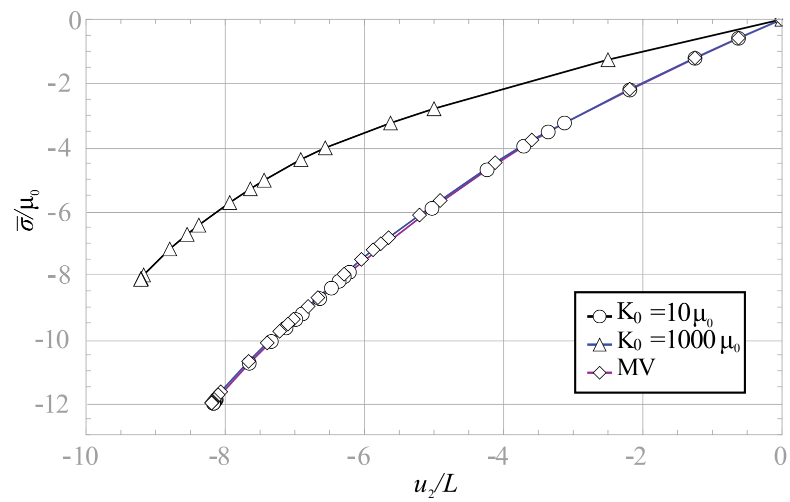


**Fig. 11** Plot of averaged stress $\bar{\sigma}$ as a function the displacement $u_2$

### *5.2 Axially compressed elliptical tube*

Next presented example is a problem of axially compressed elliptical thick-walled tube with dimensions: $R_1 = 500$ mm, $R_2 = 0.6R_1$, wall thickness $t = 0.1R_2$ and length $h = 20R_1$, see Fig. 12. Finite element model in ABAQUS/Standard is defined by 7968 8-node linear brick elements (C3D8) for compressible behaviour and with hybrid finite element formulation for incompressible and nearly incompressible behaviour (C3D8H), with three elements in thickness direction. Similarly to the previous example, the MCIZ with $K_0 = 10\mu_0$ and $K_0 = 1000\mu_0$ is considered here. The initial shear modulus and other parameters take the values: $\mu_0 = 0.3$, $f = 0.75$, $c = 0.1$. Additionally, the problem is solved assuming the incompressible MIZ model. A modified Riks algorithm is used to obtain the solution (Dassault Systèmes, 2015). Prescribed displacement is set as follows: at the B end of the tube $u_1 = u_2 = 0, u_3 = 6R_1$, at the A end of the tube $u_1 = u_2 = u_3 = 0$, see Fig. 12.

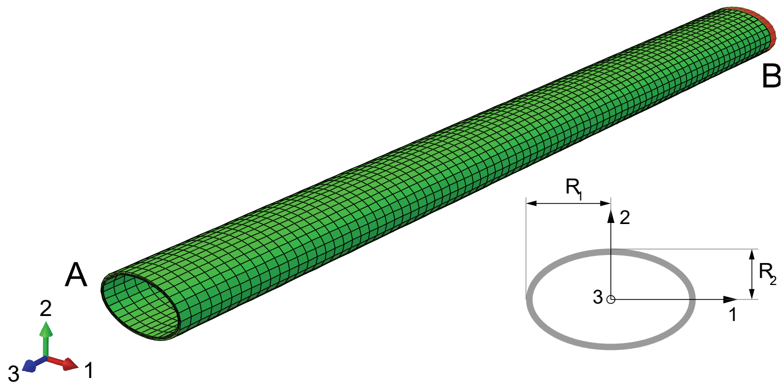


**Fig. 12** Cross-section geometry and FEM mesh

**Fig. 13** shows plot of averaged stress $\bar{\sigma}$ as a function of displacement of the B end of the tube. The averaged stress values are obtained by averaging values of the component $\sigma_{33}$ from the Gauss points of region marked in red. In addition, three deformed configurations are presented with respect to marked points. Each colour represents particular set of material parameters: blue – incompressible material (MIZ), red and black being – slightly compressible materials (MCIZ) with $K_0 = 10\mu_0$ and $K_0 = 1000\mu_0$, respectively. For the incompressible material full load range is achieved. In contrast, for nearly incompressible material maximum displacement reaches $u_3 \approx 0.1R_1$. In obtained solutions it can be noted that loss of stability is of a global nature.

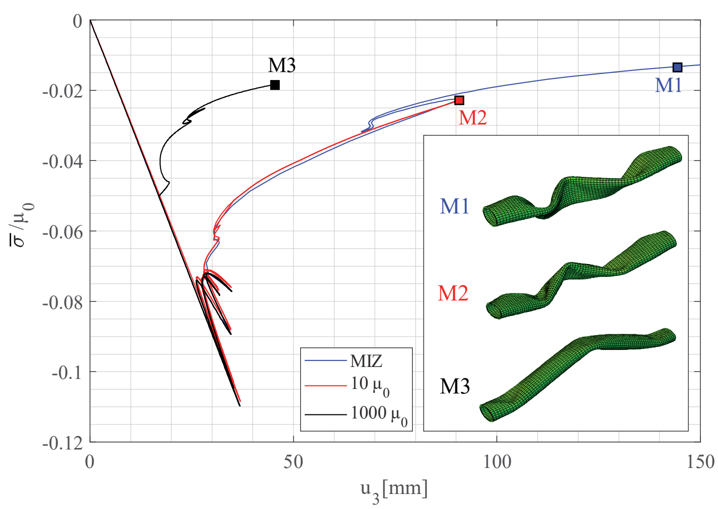


**Fig. 13.** Plots of averaged stress   as a function of tube end displacement during compression

## 6. Conclusion

The work was intended as an attempt to motivate an application of a proper generalization certain of incompressible, hyperelastic material models, namely MV and MIZ models. The generalization is provided by extending basic models to capture slight compressibility of a material. The approach leads to clear physical interpretation of a response of a material, i.e. volumetric and isochoric parts of deformation are decoupled. The description is especially suitable for rubber-like material, because of their nearly incompressibility. Moreover, the approach significantly simplifies the interpretation of experimental results for such materials.

As the ABAQUS's library of material models is limited, it provides number of ways to implement user's own model. In the case of isotropic hyperelasticity, the easies one is the UHYPER user subroutine. In the paper we provide detailed description how to implement one of the considered slightly compressible material model. The implementation is verified by basic numerical tests. Obtained results are compared to values following from the presented analytical formulas. Moreover, a Fortran code of the UHYPER is presented in the Appendix. We believe that the paper helps to understand material modelling in the case of hyperelastic, slightly compressible materials.

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## Appendix

Fortran code (free form) of the UHYPER user subroutine for the MCIZ material model

```fortran
subroutine uhyper(bi1,bi2,aj,u,ui1,ui2,ui3,temp,noel,cmname,incmpflag,numstatev,statev,numfieldv,fieldv,fieldvinc,numprops,props)
include 'aba_param.inc'
character*80 cmname
dimension u(2),ui1(3),ui2(6),ui3(6),statev(*),fieldv(*),fieldvinc(*),props(*)

!Parameters of the model
amu0=0.3
f=0.3
c=0.1
ak0=1000.0*amu0

!Stored-energy function
u(1)=0.5*amu0*(f*(bi1 - 3.) + (1. - f)*(bi2 - 3.) + 0.5*c*(bi1 - 3.)**2) + ak0*((aj**2 - 1.)/4. - Log(aj)/2.)

!First order derivatives
ui1(1)=(((-3. + bi1)*c + f)*amu0)/2.
ui1(2)=-((-1. + f)*amu0)/2.
ui1(3)=((-1. + aj**2)*ak0)/(2.*aj)

!Second order derivatives
ui2(1)=(c*amu0)/2.
ui2(2)=0.
ui2(3)=((1. + aj**2)*ak0)/(2.*aj**2)
ui2(4)=0.
ui2(5)=0.
ui2(6)=0.

!Third order derivatives
ui3(1)=0.
ui3(2)=0.
ui3(3)=0.
ui3(4)=0.
ui3(5)=0.
ui3(6)=-(ak0/aj**3)
return
```