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Retracted: Using the method of differential equations by quadratic to solve the free vibrations of columns under the effect of axial load and column weight

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A B S T R A C T

This paper investigates various techniques used to solve the differential equations governing the free vibration of columns. The present work focuses on the study of the free vibration of Euler's Bernoulli column of equal strength in compression, considering its own weight and the axial load in compression and tension while subjected to symmetrical boundary conditions. The investigation utilizes the differential quadrature method to examine the fifth natural frequency parameters of the column in different states of column boundary conditions and varying geometric section shapes, including pin-pin and clamp-clamp configurations. The results of this work contribute valuable insights for informed decisions on selecting the cross-section types and appropriate boundary conditions for ensuring the stability of such columns in civil constructions.

1. Introduction

The study of stability and vibration of structures is an area of great interest in civil constructions. Most applications of engineering work draw their safety on the strength of the structural elements. Very often these structural elements are oversized due to the no uniformity of strength in the structure. This is to guarantee the stability of structural members and the structure in general. A structural element model of equal strength ensures more stability of the structure over its entire length. This should best reduce its cross-section and increase its strength which remains constant and therefore optimizes the use of the material. This model is more important because it finds applications in more field such as: telecommunication, Electrical Engineering for telecommunication and electric towers, civil engineering for pillars in building and pillars bridges, mechanical engineering for mine shaft traction cable and so on this model of equal strength leads us to a column model of variable bending stiffness, thus varying the right section of it while considering the action of gravity on it as the case of the Eiffel tower. Hosseini et al. (2014) used numerical method including Neiumero, Calculus of Variation and Finite Differences for determining critical load of columns is one of the most important factors for choosing quality type of column, length, profile, etc. before determining critical load of column, we need to form a differential equation with some boundary conditions. These equations are determined by internal and external moments and boundary conditions in accordance with type of support.

Engineering problems in general lead in their mathematical modelling after several transformations to eigenvalue problems which do not have exact solutions. In order to better solve this problem, many advanced numerical methods have often been used to propose a solution that best approaches the exact solution. Wen et al. (2017) used Laplace Transform to

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solve free vibration of continuous beams. Their paper specifies the application method of Laplace Transform in solving the free vibration problems of continuous beams, having great significance in the proper use of the transform method. The finite element method was used to investigate the vibration characteristics of uniform hanging beams under gravity (Yokoyama, 1990). They demonstrate that the natural frequencies and the associated mode shapes obtained for the classical Euler's Bernoulli beam, which is a special case of the present model, show good agreement with the available experimental and numerical results. Motaghian et al. (2018) employed the combination of Fourier sine and cosine series to develop an analytical method to conduct the free vibration analysis of Euler's Bernoulli beam with varying cross-section. The advantages of the Fourier series make this analysis easier and more extensive than any other existing solutions. Okay et al. (2010) applied the variational iteration method to determine the buckling loads and mode shapes of heavy columns under its own weight. Their study shows efficiency of the method and accuracy of the obtained analytic approximate solution. Lee and Schultz (2004) applied the Chebychev pseudospectral method to solve the vibration of Timoshenko beams and Mindlin plates. They show that the results from this method agree with those of Euler's Bernoulli beams and Kirchhoff plates when the thickness-to-length (radius) ratio is very small. However these results deviate considerably as the thickness-to-length (radius) ratio grows larger. Yagci et al. (2009) used a spectral Chebychev technique to solve linear and nonlinear beam equations. They study convergence and accuracy characteristics of the spectral-Chebychev technique by solving eigenvalue problems with different boundary conditions. It is found in their study that the convergence is exponential, and a small number of polynomials is sufficient to obtain machine-precision accuracy. The differential quadrature method was used to investigate the effect of columns on the natural frequencies and mode shapes (Mahmoud et al., 2011). In comparison with traditional techniques such as finite element and finite difference, they demonstrated that the quadrature differential method is an efficient method in solving the free vibration of non-uniform columns with good accuracy using a considerably small number of grid points. Taha and Essam (2013) used the DQM to study the stability behavior and free vibration of axially loaded tapered columns with elastic end restraints. The solutions obtained were compared to those obtained from finite element methods and found in close agreement. Many others authors used the differential quadrature method and they have presented there are effectiveness to determine both frequency parameters and buckling factors. Torabi and Afshari (2016) used the DQM for vibration analysis of cantilevered non-uniform trapezoidal thick plates.

In view of these multiple applications in different fields of engineering and the absence of stability and vibration works on this type of column of equal strength, we will be interested to investigate its vibrational behavior. We would also see the impact of the particularity of equal strength of column with variable cross-section on the vibration analysis by considering some different geometric shapes admissible in the constructions. Therefore, in the present work, we use the differential quadrature method to determine the vibration modes in the case of free vibration of the column of equal strength in traction and compression. In order to verify the accuracy of our calculation code, the case of constant cross-section with only axial load is compared with those results known in literature. Then we deal with different following cases: the case of tip force, the case of own weight and finally the coupled effect of tip force and self-weight.

The paper is organized as follows: in section 2, we present the mathematical modelling of the column concerning the variation law of the cross-section, the variation of self-load, the hypothesis of this work and the governing equation using Hamilton's principle. In section 3 we present the formulation of the quadrature differential method, where we use this approach to discretize the governing equation and the boundary conditions. Section 4 is devoted to the presentation of numerical results and discussions and in section 5 we present the conclusion of our work.

2. Problem formulation and mathematical modelling

2.1. Equal strength column modelling

The equal resistance columns have a particular cross section and quadratic moment, to the point that the law and the geometrical forms of their cross-section should be determined.

2.1.1. Law of variation of the cross-section

Let us consider a column of mass density ρ and length l, of variable section such that its stress σ is the same along its entire column. Arranged vertically along the (ox) axis and loaded axially by a load p. The hypothesis of equal resistance applied to the column allows us to write,

$$\sigma(x) = \frac{N(x)}{A(x)} \tag{1}$$

where $N(x)$ represents the normal force in the column and $A(x)$ it's cross sectional area at any altitude x.

By applying the fundamental principle of statics on the column elemental column of length dx, we have:

$$N + dN - N + \rho g A dx = 0 \tag{2}$$

$$\frac{dA}{A} = -\frac{\rho g}{\sigma} dx \tag{3}$$

$$A(x) = A_0 e^{(-\frac{\rho g}{\sigma}x)} \tag{4}$$

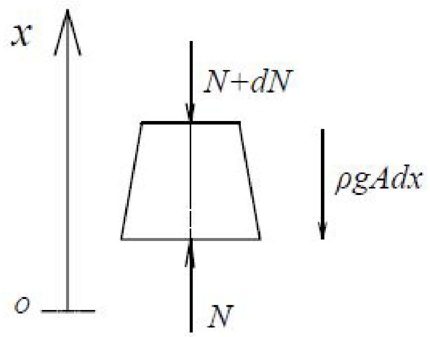


Fig. 1. Infinitesimal column elemental column.

The base section is dimensioned taking into account the axial load, A_0 representing the section at the base $x = 0$ and can still take the form ($A_0 = p/\sigma$).

In the rest of our work we will introduce a parameter m which takes into account the variation of the cross section. For this purpose we distinguish two cases, the case where the cross section is variable ($m = 1$) and the case where the cross section is constant ($m = 0$). Under this assumption the law of variation of the cross-sectional area will be written as continuation,

$$A(x) = A_0 e^{(-\frac{\rho g m}{\sigma}x)} \tag{5}$$

The self-weight of the column per unit length can be written as,

$$q(x) = q_0 e^{(\frac{\rho g m}{\sigma}x)} \tag{6}$$

where q_0 represents the weight per unit length at the base of the column at altitude $x = 0$.

2.1.2. Expression of quadratic moments for different shapes of cross-sections

The geometry of the cross-section imposes a law of variation of the quadratic moment. After calculation of quadratic moments by considering six usual cases of cross-section shape, we expressed the quadratic moments for different shapes as a function of a number n which we called here the shape factor. Thus the general form of variation of the quadratic moment can be written in the following form,

$$I(x) = I_0 e^{(-\frac{\rho g n}{\sigma}x)} \tag{7}$$

where I_0 is the basic quadratic moment. The shape factor n takes different values depending on the type of cross section. For the rectangular cross-section with variable length and constant width ($n = 1$), ($n = 3$) where the length is constant and width variable. The case $n = 2$ is for square, circle cross-section and ($n = 4$) for rhomb cross-section.

2.2. Hypothesis of study

We study a free vibration of equal strength Euler's Bernoulli beam column axially loaded with its own weight and concentrically applied tip force. The cross sections chosen are those with various shapes such as rectangular, square, circle and rhomb. We are not considering the nonlinear geometry of the cross-section of the beam. The column is straight along the longitudinal direction before the axial load is applied. The material used is homogeneous, elastic and isotropic. The axial compression is small and neglected and the failure is due to flexural buckling and vibration.

The boundary conditions of the column are symmetric, that is both pinned-pinned and both clamped-clamped.

2.3. Free vibration equation model of beam-column

The problem that we want to analyses in this paper is the Free vibration analyse of pinned-pinned and clamped-clamped equal strength column under self-weight and tip force using differential quadrature method. The equation motion of the beam can be found through the variation principle.

The strain energy of this system that we consider is the energy of elastic deformation,

$$U = \frac{1}{2}\int_{0}^{L} EI(x)(\frac{\partial^2 w}{\partial x^2})^2 dx \tag{8}$$

where $I(x)$ the quadratic moment of the cross section along the length is, E is Young's modulus and $w = w(x,t)$ is the dependent deflection of the axial position x and of the time t

The work due to the axial load and the self-weight is given by:

$$W = \frac{1}{2}\int_{0}^{L} N(x)\,(\frac{\partial w}{\partial x})^2 dx \tag{9}$$

$N(x)$ Is the normal load in the beam dependent on position x.

The kinetic energy of the system is given by:

$$T = \frac{1}{2}\int_{0}^{L} \rho A(x)(\frac{\partial w}{\partial t})^2 dx \tag{10}$$

where ρ is the mass density and $A(x)$ is the cross-sectional area depending on the location x.

We can obtained the general equation of the motion using the Hamilton's principle generalized as:

$$\int_{t_1}^{t_2} \delta(T - U + W)dt = 0 \tag{11}$$

Substituting Eqs. (8-10) into Eq. (11), and after mathematical manipulations we obtain the differential equation of motion of free vibration Euler's Bernoulli beam under self-weight and axial load as:

$$\frac{\partial}{\partial t}\left(\rho A(x)\frac{\partial w}{\partial t}\right) + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(N(x)\frac{\partial w}{\partial x}\right) = 0 \tag{12}$$

For this column the normal strength is expressed as:

$$N(x) = p + \int_{x}^{L} q(x)dx \tag{13}$$

where p axial load and $q(x)$ *is* represent the weight per volume unit.

$$w(x,t) = W(x)\varphi(t) = W(x)e^{i\omega t}, \quad EI(x) = EI_0 l(x)\,, \quad q(x) = q_0 r(x) \tag{14}$$

Introducing Eq. (14), where EI_0 and A_0 are the maximum values of flexural rigidity and cross-section area per length occurring at the base $x = 0$ and ω is the vibration frequency.

$$EI_0\frac{d^2}{dx^2}\left(l(x)\frac{d^2W}{dx^2}\right) + p(\frac{d^2W}{dx^2}) + \rho g A_0\frac{d}{dx}\left[\left(\int_{x}^{L} r(x)dx\right)\frac{dW}{dx}\right] = r(x)\omega^2\rho g A_0 W \tag{15}$$

Normalizing all length by L so that $\xi = x/L$ Eq. (15) becomes:

$$\frac{d^2}{d\xi^2}\left(l(\xi)\frac{d^2W}{d\xi^2}\right) + P\left(\frac{d^2W}{d\xi^2}\right) + Q\frac{d}{d\xi}\left[\left(\int_{\xi}^{1} r(z)dz\right)\frac{dW}{d\xi}\right] - r(\xi)\Omega^2 W = 0 \tag{16}$$

$$\Omega^2 = \frac{\omega^2\rho A_0 L^4}{EI_0}\,, \quad P = \frac{pL^2}{EI_0}\,, \quad Q = \frac{q_0 L}{EI_0} \tag{17}$$

For the beam of equal strain along the beam column that we are study here

$$l(\xi) = e^{-(\frac{\rho g n L}{\sigma}\xi)}\,, \qquad r(\xi) = e^{-(\frac{\rho g m L}{\sigma}\xi)} \tag{18}$$

where n and m are called respectively the geometry factor who depend on geometry section and variation cross-section factor.

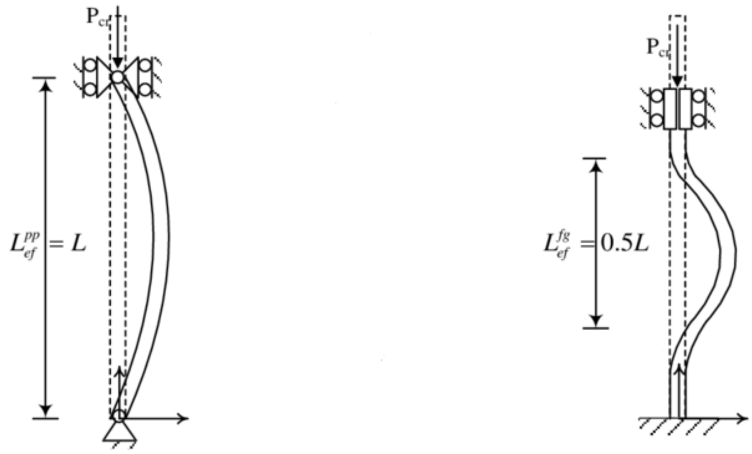


Fig. 2. Symmetric boundary ends conditions of the studied columns.

The boundary conditions are represented as follows:

For a pinned-pinned supports (P-P)

$$W = \frac{d^2W}{d\xi^2} = 0, \quad at \;\; \xi = 0 \tag{19}$$

$$W = \frac{d^2W}{d\xi^2} = 0, \quad at \;\; \xi = 1 \tag{20}$$

For a clamped-clamped supports (C-C)

$$W = \frac{dW}{d\xi} = 0, \quad at \;\; \xi = 0 \tag{21}$$

$$W = \frac{dW}{d\xi} = 0, \quad at \;\; \xi = 1 \tag{22}$$

Eq. (16) will be solved considering the boundary conditions presented by Eqs. (19) to (22) for different values of n and m.

3. Method of resolution and numerical discretization

3.1. Formulation of differential quadrature method

The differential quadrature method is a numerical discretization technique for the approximation of derivatives. Bellman et al. (1972) suggested that the first order derivative of the function $f(x)$ with respect to x at a grid point x_i is approximated by a linear sum of all the functional values in the whole domain that is,

$$f_x(x_i) = f^{(1)}(x_i) = \sum_{j=1}^{N} a_{ij} f(x_j), \quad for \;\; i = 1{:}N \tag{23}$$

where a_{ij} represent the weighting coefficients, and N is the number of grid points in the whole domain. Eq. (23) can be generalize for the rth-order derivative of $f(x)$ at the x_i point to the form,

$$f^{(r)}(x_i) = \sum_{j=1}^{N} a_{ij}^{(r)} f(x_j), \quad for \;\; i = 1{:}N \tag{24}$$

where $a_{ij}^{(r)}$ are the weighting coefficients of the rth-order derivative of the functions $f(x)$ at the grid point x_i.

To improve the Bellman et al. (1972) approaches in computing the weighting coefficients, many attempts have been made by researchers, but in this work we will choose the approach of Quan and Chang (2000) who used the following Lagrange interpolation polynomials as the test functions,

$$l_j(x) = \frac{M(x)}{(x - x_j)M^{(1)}(x_j)} , \quad j = 1{:}\,N \tag{25}$$

where

$$M(x) = \prod_{m=1}^{N} (x - x_m) \tag{26}$$

$$M^{(1)}(x) = \prod_{m=1, m\neq j}^{N} (x_j - x_m) \tag{27}$$

The weighting coefficients for the first-order derivative can be obtained using the Lagrange polynomial as follows:

$$a_{ij}^{(1)} = \left.\frac{dl_j(x)}{dx}\right|_{x=x_i} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}} , \quad j = 1{:}\,N, i \neq j \tag{28}$$

$$a_{ii}^{(1)} = \left.\frac{dl_i(x)}{dx}\right|_{x=x_i} = -\sum_{j=1, j\neq i}^{N} a_{ij}^{(1)} , \quad i = 1{:}\,N \tag{29}$$

By using the sampling points as:

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\right)\right] , \quad i = 1{:}\,N \tag{30}$$

and these coefficients give the matrix as,

$$[A^{(1)}] = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1N}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & & a_{2N}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N1}^{(1)} & a_{N2}^{(1)} & & a_{NN}^{(1)} \end{bmatrix} \tag{31}$$

In a similar manner we can evaluate the weighting coefficients of the rth-order derivative as,

$$a_{ij}^{(r)} = \left.\frac{d^r l_j(x)}{dx^r}\right|_{x=x_i} = r\left(a_{ii}^{(r-1)} a_{ij}^{(1)} - \frac{a_{ij}^{(r-1)}}{x_i - x_j}\right), \quad i,j = 1{:}\,N,\ i \neq j\ ,\ r \geq 2 \tag{32}$$

$$a_{ii}^{(r)} = \left.\frac{d^r l_i(x)}{dx^r}\right|_{x=x_i} = -\sum_{j=1, j\neq i}^{N} a_{ij}^{(r)}, \quad i,j = 1{:}\,N\ ,\ i \neq j\ ,\ r \geq 2 \tag{33}$$

This give the matrix $A^{(r)}$ which can be write as,

$$[A^{(r)}] = \begin{bmatrix} a_{11}^{(r)} & a_{12}^{(r)} & \cdots & a_{1N}^{(r)} \\ a_{21}^{(r)} & a_{22}^{(r)} & \cdots & a_{2N}^{(r)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1}^{(r)} & a_{N2}^{(r)} & \cdots & a_{NN}^{(r)} \end{bmatrix} \tag{34}$$

The weighting coefficients of the different order derivative have a relation between them such that we have,

$$a_{ij}^{(r)} = \sum_{k=1}^{N} a_{ik}^{(r-1)} a_{kj}^{(1)} , \quad i,j = 1{:}\,N\ ,\ i \neq j\ ,\ r \geq 2 \tag{35}$$

We can also build a matrix from Eq. (35),

$$[A^{(r)}] = [A^{(r-1)}][A^{(1)}] = [A^{(1)}][A^{(r-1)}] \tag{36}$$

3.2. LDQM Formulation of the governing differential equation

Eq. (16) can be written as,

$$\frac{d^2}{d\xi^2}\left(l(\xi)\frac{d^2W}{d\xi^2}\right) + P\left(\frac{d^2W}{d\xi^2}\right) + Q\frac{d}{d\xi}\left[\left(\int_\xi^1 r(z)dz\right)\frac{dW}{d\xi}\right] - r(\xi)\Omega^2 W = 0 \tag{37}$$

We can discretize Eq. (37) using the (LDQM) as

$$\frac{I(\xi_i)}{r(\xi_i)}\sum_{k=1}^{N} a_{ij}^{(4)} W(\xi_j) + \frac{2}{r(\xi)}\frac{dI(\xi)}{d\xi}\bigg|_{\xi=\xi_i} \sum_{k=1}^{N} a_{ij}^{(3)} W(\xi_j) + \left(\frac{1}{r(\xi)}\frac{d^2 I(\xi)}{d\xi^2} + \frac{P}{r(\xi)} + \frac{Q}{r(\xi)}\left(\int_{\xi}^{1} r(z)dz\right)_{\xi=\xi_i}\right)\sum_{k=1}^{N} a_{ij}^{(2)} W(\xi_j) - Q\sum_{k=1}^{N} a_{ij}^{(1)} W(\xi_j) - \Omega^2 W(\xi_i) = 0\,, \quad i = 1{:}N \tag{38}$$

We can also discretize the boundary conditions given by Eqs. (19) to (22) as,

Clamped-Clamped support (C-C):

$$W(\xi_1) = W(\xi_N) = 0\,,\ \sum_{k=1}^{N} a_{1j}^{(1)} W(\xi_j) = \sum_{k=1}^{N} a_{Nj}^{(1)} W(\xi_j) = 0 \tag{39}$$

Pinned-Pinned support (P-P):

$$W(\xi_1) = W(\xi_N) = 0\,,\ \sum_{k=1}^{N} a_{1j}^{(2)} W(\xi_j) = \sum_{k=1}^{N} a_{Nj}^{(2)} W(\xi_j) = 0 \tag{40}$$

4. Numerical results and discussions

In this section we will use the LDQM to investigate the vibration of non-uniform columns under tip force and self-weight. First of all free vibration of homogeneous uniform columns who have an exact solution is investigated to show the effectiveness of the method.

Then the effect of the cross-section is investigated in different boundary conditions.

4.1. Uniform homogeneous column

For this case Eq. (16) takes the simple form,

$$\frac{d^4W}{d\xi^4} + P\frac{d^2W}{d\xi^2} - \Omega^2 W = 0 \tag{41}$$

The exact solution can be solved taking $W(\xi) = Ae^{\lambda\xi}$ by substituting this expression in Eq. (41), we obtain the characteristic equation

$$\lambda^4 + P\lambda^2 - \Omega^2 = 0 \tag{42}$$

The solution of this ordinary differential equation is presented as

$$W(\xi) = A_1\cosh(\alpha\xi) + A_2\sinh(\alpha\xi) + A_3\cos(\beta\xi) + A_4\sin(\beta\xi) \tag{43}$$

where α and β are given by:

$$\alpha = \sqrt{\frac{\sqrt{P^2+4\Omega^2}-P}{2}}\ , \qquad \beta = \sqrt{\frac{\sqrt{P^2+4\Omega^2}+P}{2}} \tag{44}$$

Then the constants A_1, A_2, A_3, A_4 can be determined using the boundary conditions.

4.1.1. Exact frequencies parameters

For the case of pinned-pinned boundary conditions we will use Eq. (19) and Eq. (20) to have the following system:

$$\begin{cases} A_1 + A_3 = 0 \\ \alpha^2 A_1 - \beta^2 A_3 = 0 \\ A_1\cosh\alpha + A_2\sinh\alpha + A_3\cos\beta + A_4\sin\beta = 0 \\ \alpha^2 A_1\cosh\alpha + \alpha^2 A_2\sinh\alpha - \beta^2 A_3\cos\beta - \beta^2 A_4\sin\beta = 0 \end{cases} \tag{45}$$

We can put the above systems of Eqs. (45) in the matrix form as:

$$\begin{bmatrix} \sinh\alpha & \sin\beta \\ \alpha^2\sinh\alpha & -\beta^2\sin\beta \end{bmatrix}\begin{Bmatrix} A_2 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{46}$$

which is simply

$$[M]\{A\} = \{0\} \tag{47}$$

The trivial solution of this homogeneous system of Eq. (46) permit us to compute $det(M) = 0$

$$-(\alpha^2 + \beta^2)\sinh\alpha\sin\beta = 0 \quad (48)$$

Since $\sinh(\alpha) \geq 0$ for all values of, the only roots to this equation are,

$$\sin\beta = 0 \rightarrow \beta = k\pi, \qquad k = \{0,1,\dots,n\} \quad (49)$$

The general frequencies parameters for free vibration beam column under self-weight given is Eq. (50), where k is the vibration mode:

$$\Omega^2 = -\frac{P^2}{4} + \frac{(2k^2\pi^2 - P)^2}{4}\ , \ \ k = \{0,1,\dots,n\} \quad (50)$$

Table 1. Vibration mode of beam column with Pinned-Pinned condition.

P	-10		-5		0		5		9.8696	
Ω	**LDQM**	**Exact**	**LDQM**	**Exact**	**LDQM**	**Exact**	**LDQM**	**Exact**	**LDQM**	**Exact**
Ω_1	14.004	14.004	12.114	12.114	9.8696	9.8696	6.9326	6.9326	0.0064	0.0064
Ω_2	44.196	44.196	41.904	41.904	39.478	39.478	36.894	36.894	34.189	34.189
Ω_3	93.693	93.693	91.292	91.292	88.826	88.826	86.29	86.29	83.746	83.746
Ω_4	16.28	16.28	160.39	160.39	157.91	157.91	155.39	155.39	152.9	152.9
Ω_5	251.69	251.69	249.23	249.23	246.74	246.74	244.23	244.23	241.75	241.75

For clamped-clamped boundary condition we can also use relations (46), to get:

$$\begin{cases} A_1 + A_3 = 0 \\ \alpha A_2 + \beta A_4 = 0 \\ A_1\cosh\alpha + A_2\sinh\alpha + A_3\cos\beta + A_4\sin\beta = 0 \\ \alpha A_1\sinh\alpha + \alpha A_2\cosh\alpha - \beta A_3\sin\beta + \beta A_4\cos\beta = 0 \end{cases} \quad (51)$$

This help us to build the homogeneous system as follow:

$$\begin{bmatrix} \beta(cosh\alpha - cos\beta) & \beta sinh\alpha - \alpha sin\beta \\ \alpha sinh\alpha + \beta sin\beta & \alpha(cosh\alpha - cos\beta) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (52)$$

The determinant of the new system (52) that gives the relation (53) can be solved numerically.

$$\det[M] = 0 \rightarrow 2\alpha\beta(1 - cos\beta.cosh\alpha) + (\alpha^2 - \beta^2)sin\beta.sinh\alpha = 0 \quad (53)$$

We note that for traction and compression cases corresponding respectively to the negative and positive axial load values and for the pinned-pinned and clamped-clamped boundary conditions, the differential quadrature method gives the same results as the exacts results respectively for pinned-pinned and clamped-clamped boundary conditions showed in **Table 1**, and **Table 2**, respectively. As shown previously, for very small iterations which shows the effectiveness of the results and the convergence of the method in solving this problem.

Table 2. Vibration mode of beam column with Clamped-Clamped condition.

P	-40		-20		0		20		39.478	
Ω	**LDQM**	**Exact**	**LDQM**	**Exact**	**LDQM**	**Exact**	**LDQM**	**Exact**	**LDQM**	**Exact**
Ω_1	31.347	31.347	27.274	27.274	22.373	22.373	15.848	15.848	0	0
Ω_2	75.040	75.040	68.708	68.708	61.672	61.672	53.65	53.65	44.364	44.364
Ω_3	136.26	136.26	128.82	128.82	120.90	120.90	112.42	112.42	103.48	103.48
Ω_4	216.34	216.34	208.27	208.27	199.86	199.86	191.08	191.08	182.12	182.12
Ω_5	315.74	315.74	307.27	307.27	298.56	298.56	289.58	289.58	280.56	280.56

4.2. Non-uniform homogeneous column

In the case studied, the column has the same resistance in compression along the longitudinal axis, hence we have an exponential variation of cross section area and quadratic moment as presented above.

4.2.1. Effect of tip force on frequency parameter

Here we analyze the vibration of the column in absence of its own weight. We have investigated the effect of tip force on the free vibration of the column by drawing out the frequency parameter for the first five vibration modes by varying normalized axial load that we have presented in **Table 3** and **Table 4**.

We have distinguished two cases of axial load, traction and compression corresponding respectively to negative and positive values of load. We found that the frequency value for the first vibration modes decreases until cancelled to a value of the parameter P corresponding to the critical value, which is the value for which the frequency parameter is zero.

Table 3. Vibration mode of beam column with Pinned-Pinned condition.

	$n = 1, m = 1$					$n = 2, m = 1$				
P	-10	-5	0	5	9.8696	-10	-5	0	5	9.8696
Ω_1	14.002	12.113	9.869	6.9346	0.2361	14.0	12.11	9.8668	6.9306	0.006
Ω_2	44.194	41.903	39.478	36.895	34.193	44.184	41.892	39.467	36.883	34.180
Ω_3	93.690	91.291	88.826	86.292	83.749	93.666	91.266	88.801	86.266	83.722
Ω_4	162.83	160.39	157.91	155.40	152.90	162.79	160.35	157.87	155.35	152.86
Ω_5	251.69	249.23	246.74	244.23	241.76	251.62	249.16	246.67	244.16	241.69

	$n = 3, m = 1$					$n = 4, m = 1$				
P	-10	-5	0	5	9.8696	-10	-5	0	5	9.8696
Ω_1	13.998	12.109	9.864	6.9266	0	13.996	12.106	9.8611	6.9226	0
Ω_2	44.174	41.881	39.456	36.871	34.166	44.164	41.871	39.445	36.859	34.153
Ω_3	93.642	91.241	88.776	86.239	83.695	93.618	91.217	88.750	86.213	83.669
Ω_4	162.75	160.30	157.82	155.30	152.81	162.70	160.26	157.78	155.26	152.76
Ω_5	251.55	249.09	246.6	244.09	241.61	251.48	249.02	246.56	244.02	241.54

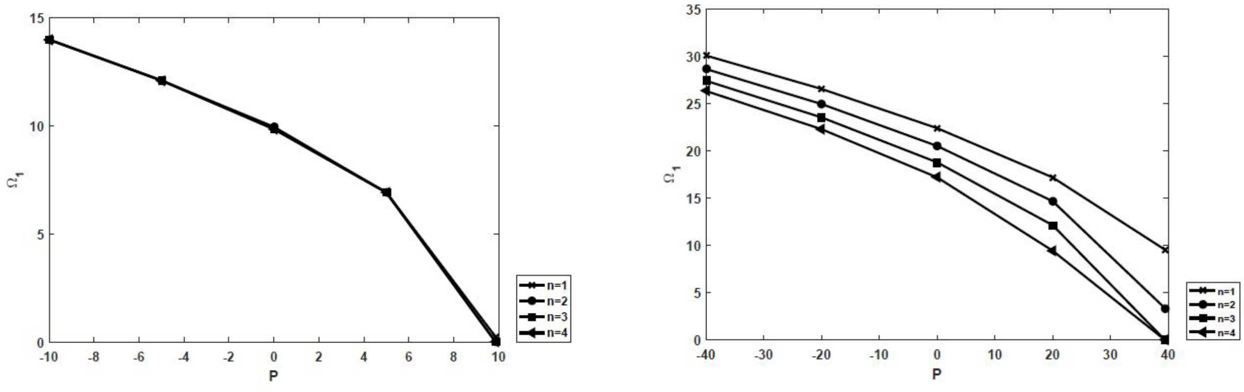
Table 4. Vibration mode of beam column with Clamped-Clamped condition.

	$n = 1, m = 1$					$n = 2, m = 1$				
P	-40	-20	0	20	39.478	-40	-20	0	20	39.478
Ω_1	30.059	26.531	22.391	17.186	9.5356	28.637	24.938	20.508	14.674	3.349
Ω_2	73.053	67.633	61.697	55.069	47.659	68.615	62.849	56.448	49.149	40.693
Ω_3	133.92	127.60	120.93	113.86	106.52	124.60	117.81	110.59	102.85	94.702
Ω_4	213.80	206.96	199.89	192.55	185.11	197.80	190.43	182.76	174.74	166.56
Ω_5	313.07	305.92	298.58	291.06	283.55	288.66	280.93	272.97	264.77	256.53

	$n = 3, m = 1$					$n = 4, m = 1$				
P	-40	-20	0	20	39.478	-40	-20	0	20	39.478
Ω_1	27.396	23.525	18.779	12.137	0	26.321	22.278	17.192	9.441	0
Ω_2	64.659	58.525	51.604	43.503	33.620	61.159	54.636	47.139	38.062	26.055
Ω_3	116.17	108.87	101.03	92.510	83.367	108.60	100.75	92.204	82.753	72.362
Ω_4	183.23	175.27	166.91	158.10	149.03	170.03	161.39	152.26	142.53	132.36
Ω_5	266.32	257.93	249.25	240.26	231.16	245.96	236.83	227.32	217.40	207.28

For the case of pinned-pinned boundary conditions, we discover that the values of frequency parameters vary slowly for different geometrical sections, but offer maximal values for the case where the geometric parameter ($n = 1$).

In the case of clamped-clamped boundary condition we have also evaluated the values of the parameter P in traction and compression, we note here that the frequency parameter admits larger values and decreases in a more accelerated way than in the case of pinned-pinned boundary condition. Thus one can say that the pinned-pinned column is less sensitive in terms of vibration frequency variation than the clamped-clamped one. Fig. (3) Shows more the variation of natural frequency parameters with the tip force.



(a) (b)

Fig. 3. First vibration frequency parameter Ω_1 under tip force for both pinned and both clamped end condition respectively to (a) and (b).

4.2.2. Effect of self-weight on frequencies parameters

In this second section, we investigate the effect of self-weight on the vibration of the column in the absence of tip force shown in **Table 5**, **Table 6** and clearly presented in **Fig. 4**.

Table 5. Vibration mode of beam column under self-weight with pinned-pinned condition.

$n = 1, m = 1$

Q	0	2.5	5	7.5	10	12.5	15	17.5	18.5729
Ω_1	9.869	9.2189	8.5078	7.7188	6.8235	5.7713	4.4496	2.4516	0.0169
Ω_2	39.478	38.848	38.205	37.549	36.88	36.197	35.50	34.787	34.476
Ω_3	88.826	88.199	87.566	86.928	86.285	85.639	84.981	84.321	84.036
Ω_4	157.91	157.29	156.66	156.03	155.39	154.36	154.11	153.47	153.47
Ω_5	246.74	246.12	245.49	244.86	244.23	243.59	242.96	242.32	242.05

$n = 2, m = 1$

Q	0	2.5	5	7.5	10	12.5	15	17.5	18.5643
Ω_1	9.8668	9.2159	8.5047	7.7154	6.8198	5.7669	4.444	2.4416	0.00731
Ω_2	39.467	38.836	38.193	37.5383	36.869	36.186	35.488	34.775	34.467
Ω_3	88.801	88.174	87.541	86.903	86.259	85.610	84.955	84.295	84.01
Ω_4	157.87	157.24	156.61	155.98	155.35	154.71	154.07	153.42	153.15
Ω_5	246.67	246.04	245.42	244.79	244.16	243.52	242.89	242.25	241.98

$n = 3, m = 1$

Q	0	2.5	5	7.5	10	12.5	15	17.5	18.5729
Ω_1	9.864	9.213	8.501	7.712	6.816	5.7626	4.4385	2.4317	0.0658
Ω_2	39.456	38.825	38.182	37.526	36.857	36.174	35.476	34.763	34.457
Ω_3	88.776	88.148	87.51	86.877	86.233	85.584	84.929	84.269	83.988
Ω_4	157.82	157.20	156.57	156.94	155.30	154.66	154.02	153.38	153.10
Ω_5	246.60	245.97	245.35	244.08	244.08	243.45	242.82	242.18	241.95

$n = 4, m = 1$

Q	0	2.5	5	7.5	10	12.5	15	17.5	18.5643
Ω_1	9.861	9.210	8.498	7.708	6.812	5.7582	4.432	2.4216	0.0323
Ω_2	39.445	38.814	38.170	37.514	36.845	36.162	35.464	34.751	34.447
Ω_3	88.750	88.123	87.490	86.851	86.208	85.558	84.903	84.243	83.964
Ω_4	157.78	157.15	156.52	156.89	155.25	154.62	153.97	153.33	153.06
Ω_5	246.53	245.90	245.28	244.65	244.01	243.38	242.75	242.11	241.84

We find that the vibration frequencies decrease when the self-weight of the column increases until it reaches a certain critical load beyond which the system would become unstable. In the case of pinned-pinned boundary conditions, it is also noted that the vibration frequencies are varying slowly for each type of cross section, which does not clearly show the effect of variation of the cross section as shown in **Table 5.** Whereas in the case of bi-clamped boundary conditions the variation of frequency parameters is more observing for different cross sections and also the critical load values are different and taking the greatest value for the case of cross section $n = 1$ as shown in **Table 6**.

Table 6. Vibration mode of beam column under self-weight with clamped-clamped condition.

$n = 1, m = 1$

Q	0	5	10	15	20	25	30	35
Ω_1	22.391	20.70	18.850	16.787	14.421	11.570	7.719	0.006
Ω_2	61.697	59.484	57.178	54.765	52.23	49.553	46.707	43.655
Ω_3	120.93	118.54	116.09	113.59	111.04	108.42	105.73	102.98
Ω_4	199.89	197.39	194.29	192.29	189.69	187.05	184.38	181.66
Ω_5	298.58	296.02	293.43	290.82	288.18	285.52	282.83	280.12

$n = 2, m = 1$

Q	0	5	10	15	20	24	26	28.12
Ω_1	20.508	18.625	16.515	14.075	11.09	7.914	5.691	0.0411
Ω_2	56.448	53.992	51.408	48.672	45.756	43.267	41.961	40.525
Ω_3	110.59	107.94	105.22	102.42	99.538	97.170	95.964	94.669
Ω_4	182.76	177.18	177.18	174.32	171.41	169.04	167.85	166.57
Ω_5	272.97	267.25	267.25	264.34	261.41	259.03	257.83	256.56

$n = 3, m = 1$

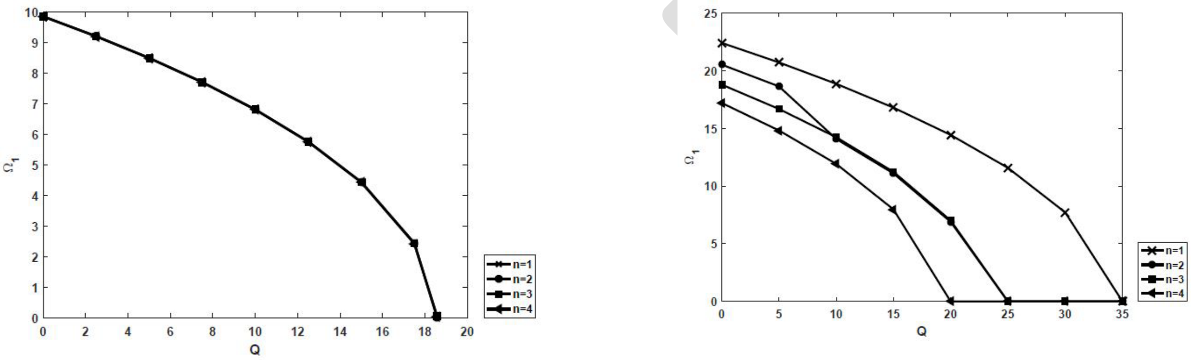
Q	0	5	10	15	20	22	23	25
Ω_1	18.779	16.669	14.225	11.230	7.013	4.271	1.658	0
Ω_2	51.604	48.865	45.944	42.802	39.381	37.160	37.160	35.593
Ω_3	101.03	98.076	95.024	91.865	88.588	87.242	86.560	85.181
Ω_4	166.91	163.83	160.68	157.47	154.19	152.86	152.19	150.84
Ω_5	249.25	246.09	242.88	239.62	236.32	234.99	234.32	232.98

$n = 4, m = 1$

Q	0	5	10	15	17	18	19	20
Ω_1	17.191	14.809	11.917	7.974	5.648	4	0.224	0.0068
Ω_2	47.139	44.065	40.732	37.061	35.474	34.650	33.804	32.933
Ω_3	92.204	88.894	85.447	81.846	80.358	79.604	78.841	78.071
Ω_4	152.26	148.81	145.27	141.64	140.16	139.41	138.66	137.91
Ω_5	227.32	223.78	220.18	216.51	215.03	214.28	213.53	212.78

4.2.3. Coupled effect of self-weight and axial force on frequencies parameters

We have investigated the effect of frequency parameter on free vibration for various tip load and fixed self-weight as presented in **Table 7** and **Table 8**.



(a) (b)

Fig. 4. First frequency parameter Ω_1 under self-weight for both pinned and clamped ends conditions respectively to (a) and (b).

We also note that for pinned-pinned boundary conditions the self-frequencies are varying slowly with the cross section in the same order of magnitude, but in the case of clamped-clamped boundary conditions this variation is better observed. We can say that this behavior is due to the assumption of equal resistance assumed above. We also note that for small values of self-weight of the column in traction, the self-frequencies are higher and decrease as the positive values of tip force parameter increases. For cases of compression the proper frequencies of vibration decrease very quickly until it's cancelled. For the geometrical cross section ($n = 4$) the frequencies parameters give smallest values. The case where cross section parameter ($n = 1$) the frequencies parameters are giving highest values.

Table 7. Vibration mode of beam column under self-weight and axial load with bi-Pinned condition.

$n = 1, m = 1$

	Q=5			**Q=10**			**Q=15**			**Q=20**		
P	-10	-5	9.8696	-10	-5	5	-10	-5	5	-10	-5	5
Ω₁	13.084	11.036	0	12.066	9.802	0	10.920	8.340	0	9.593	6.485	0
Ω₂	43.065	40.708	32.708	41.894	39.467	34.099	40.682	38.178	32.603	39.426	36.839	31.032
Ω₃	92.502	90.068	82.406	91.290	88.823	83.671	90.058	87.557	82.326	88.807	86.269	80.956
Ω₄	161.62	159.16	151.60	160.39	157.91	152.83	159.16	156.65	151.53	157.91	155.38	150.21
Ω₅	250.47	247.99	240.47	249.23	246.74	241.69	247.99	245.49	240.40	246.74	244.22	239.12

$n = 2, m = 1$

	Q=5			**Q=10**			**Q=15**			**Q=20**		
P	-10	-5	7.3177	-10	-5	5	-10	-5	5	-10	-5	5
Ω₁	13.082	11.034	0.024	12.064	9.799	0.089	10.917	8.337	0.0939	9.591	6.481	0.0487
Ω₂	43.055	40.697	34.201	41.883	39.456	34.265	40.671	38.167	34.369	39.416	36.828	34.512
Ω₃	92.478	90.043	83.744	91.265	88.798	83.808	90.034	87.531	83.913	88.782	86.244	84.058
Ω₄	161.58	159.12	152.88	160.35	157.87	152.94	159.11	156.61	153.05	157.86	155.34	153.19
Ω₅	250.40	247.92	241.71	249.16	246.67	241.77	247.92	245.42	241.88	246.64	244.14	242.02

$n = 3, m = 1$

	Q=5			**Q=10**			**Q=15**			**Q=20**		
P	-10	-5	7.3124	-10	-5	5	-10	-5	5	-10	-5	-0.8146
Ω₁	13.080	11.031	0.0043	12.062	9.797	0.0828	10.915	8.334	0.007	9.588	6.477	0.0223
Ω₂	43.045	40.686	34.191	41.873	39.445	34.255	40.661	38.156	34.359	39.405	36.817	34.502
Ω₃	92.454	90.018	83.720	91.241	88.773	83.784	90.009	87.506	83.889	88.757	86.218	84.034
Ω₄	161.53	159.07	152.83	160.31	157.82	152.90	159.07	156.56	153.00	157.82	155.29	153.15
Ω₅	250.33	247.85	241.64	249.09	246.60	241.70	247.85	245.34	241.81	246.60	244.08	241.96

$n = 4, m = 1$

	Q=5			**Q=10**			**Q=15**			**Q=20**		
P	-10	-5	7.307	-10	-5	4.681	-10	-5	1.972	-10	-5	-0.82
Ω₁	13.078	11.029	0.0217	12.060	9.794	0.0755	10.913	8.331	0.0162	9.586	6.473	0.081
Ω₂	43.034	40.675	34.181	41.863	39.434	34.245	40.650	38.145	34.349	39.394	36.805	34.493
Ω₃	92.429	89.993	83.696	91.217	88.748	83.760	89.985	87.481	83.865	88.733	86.193	84.011
Ω₄	161.49	159.03	152.79	160.26	157.78	152.85	159.02	156.52	153.96	157.77	155.29	153.11
Ω₅	250.26	247.78	241.57	249.02	246.53	241.63	247.78	245.27	241.74	246.53	244.01	241.89

Table 8. Vibration mode of beam column under self-weight and axial load with bi-Clamped condition.

$n = 1, m = 1$

	Q=5			Q=10			Q=15			Q=20		
P	-40	20	41	-40	20	33.9	-40	20	30	-40	10	19.9
Ω1	28.873	14.843	0.007	27.633	12.030	0.004	26.333	8.280	0.006	24.964	20.443	0.009
Ω2	71.215	52.561	44.040	69.324	49.917	44.121	67.373	47.114	42.743	65.358	48.354	44.162
Ω3	131.77	111.31	103.18	129.58	108.71	103.26	127.35	106.03	102.04	125.07	107.23	103.33
Ω4	211.47	189.95	181.81	209.11	187.32	181.90	206.72	184.65	180.71	204.30	185.86	181.98
Ω5	310.63	288.43	280.24	300.89	285.78	280.33	305.67	283.09	279.15	303.17	284.31	280.42

$n = 2, m = 1$

	Q=5			Q=10			Q=15			Q=20		
P	-40	20	33.9	-40	20	25.85	-40	10	18.485	-40	10	15.35
Ω1	27.384	11.784	0.011	26.068	7.846	0.009	24.678	9.577	0.014	23.203	3.854	0.010
Ω2	66.638	46.284	39.993	64.593	43.214	40.483	62.473	44.510	40.624	60.269	41.284	38.668
Ω3	122.25	99.990	94.073	119.86	97.047	94.524	117.42	98.302	94.670	114.92	95.299	92.952
Ω4	195.25	171.85	165.95	192.66	168.90	166.40	190.04	170.16	166.55	187.37	167.18	164.87
Ω5	285.98	261.83	255.91	283.27	258.87	256.37	280.53	260.14	256.52	277.76	257.15	254.85

$n = 3, m = 1$

	Q=5			Q=10			Q=15			Q=20		
P	-40	20	26.425	-40	10	18.965	-40	10	13.58	-40	10	18.95
Ω1	26.076	8.295	0.414	24.682	9.835	0.487	26.199	4.277	0.002	24.682	9.835	0.632
Ω2	62.534	40.178	36.925	60.324	41.519	37.076	58.017	37.991	36.104	60.324	41.519	37.084
Ω3	113.62	89.269	86.247	111.01	90.579	86.398	108.33	87.258	85.549	111.01	90.579	86.405
Ω4	180.43	154.84	151.85	177.59	156.16	152.00	174.69	152.86	151.17	177.59	156.16	152.01
Ω5	263.36	236.97	233.97	260.37	238.30	234.12	257.34	234.98	233.30	260.37	238.30	234.13

$n = 4, m = 1$

	Q=5			Q=10			Q=15			Q=20		
P	-40	10	20.75	-40	10	14	-40	-20	5.569	-40	-5	5
Ω1	24.936	10.709	0.013	23.463	5.890	0.004	21.884	16.594	1.61	20.176	6.288	0.004
Ω2	58.879	39.429	33.687	56.491	35.633	33.363	53.981	46.352	34.002	51.327	35.787	29.794
Ω3	105.83	84.114	78.648	102.97	80.459	78.375	100.02	91.396	78.985	96.970	80.679	75.375
Ω4	166.96	143.91	138.45	163.82	140.25	138.18	160.62	151.43	138.79	157.34	140.49	135.28
Ω5	242.69	218.79	213.30	239.38	215.11	213.04	236.01	226.48	213.66	232.60	215.36	210.17

5. Conclusion

Columns of variable cross sections are examples of structures that offer very high strength and guarantee high stability of the structure due to the fact that the resistance is equal along its axial straight line. The application of the principle of equal resistance yields an exponentially decreasing cross section. By applying the variation principle we established the equations of motion and then we used the differential quadrature method to discretize the problem. We have considered the case of constant cross section axially loaded that the results are very well known which will allow us to verify the accuracy of our calculation code. In the solution of this vibration problem we distinguished 03 cases of loading corresponding to tip force, self-weight and the coupling of the tip force and self-weight respectively. For each case studied, we observe that the frequency parameters of the columns vary slowly in the same order of magnitude for different cases of cross section and boundary conditions, but for the case of clamped-clamped columns, this variation is more sensible according to the highest values of self-frequencies.We can say that this behavior is due to the assumption of equal resistance assumed above. In addition, the rectangular cross section (n = 1) admits greater values of frequencies which further guarantees the stability of the column and the rhomb cross section ($n = 4$). They give the smallest values which correspond to the poor stability. This behavior justifies the choice of clamped-clamped rectangular column for the support of pillars in building construction.

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