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Ranking fuzzy numbers by volume of solid of revolution of membership function about axis of support

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A B S T R A C T

It is admissible that fuzzy numbers (FNs) are apt for representing imprecise or vague data in real-world problems. While using FNs in decision-making problems, selecting the best alternative among available alternatives is challenging, and therefore, ranking FNs is essential. We can find different studies in the literature, but to our knowledge, no one attempted to rank FNs using the concept of volume. This paper proposes a new method for ranking generalized fuzzy numbers (GFNs) using the volume of the solid obtained by revolving its membership function (MF) about the x-axis. We calculate the volumes of positive and negative sides along with the centroid of a generalized fuzzy number(GFN) to define the fuzzy number(FN) score. This score represents the defuzzified value of FN, is used to select the best alternative, and overcomes the limitations in some existing methods like ranking FNs having the same centroid, crisp numbers, symmetric fuzzy numbers, and FNs with the same core.

1. Introduction

The fuzzy set theory is useful for addressing non-stochastic impreciseness or vague concepts. It was introduced by (Zadeh, 1965) to deal with data involving uncertainty and ambiguity. Fuzzy sets can reduce uncertainty and are used to solve problems in various areas such as supply chain management, agriculture, transport, and medicine. The ranking of FNs is crucial to the fuzzy decision process because it helps to identify the best alternative from a particular set of alternatives. Jain (1976) introduced the idea of ranking FNs, which assists in problem-solving in data analysis, artificial intelligence, optimization, reasoning, and forecasting.

Defuzzification procedures for ranking FNs gained importance in the late 90s, specifically, defuzzification using centroids. To mention a few, Cheng (1998) introduced a ranking technique based on the distance method, which computes the distance of FN from origin to the centroid point, Chu and Tsao (2002) used area between centroid point of FN and origin to rank FNs, Wang and Lee (2008) suggested an updated approach of ranking FNs based on area between the centroid and the original points of an FN and further, one can see many works on centroids published in the 2000s. Later, Yao and Wu (2000) suggested a novel approach for ranking FNs based on the decomposition principle, sign distance approach was proposed for ranking FNs by Abbasbandy and Asady (2006), Asady and Zendehnam (2007) proposed a novel method for ranking FNs based on distance minimization. Abbasbandy and Hajjari (2009) proposed to rank FNs based on their left and right spreads, Chen and Chen (2009) proposed to rank FNs according to their heights and spreads, the improved distance minimization approach for ranking FNs was proposed by Asady (2011), Nejad and Maschinchi (2011) presented a novel FN ranking approach on regions of the left and right sides utilizing the deviation degree method, which can successfully rank various FNs and their images. Later, Chen et al. (2012) proposed a novel ranking algorithm for ranking GFNs with varying left and right heights, Yu et al. (2013) suggested an epsilon deviation-based ranking function, (Eslamipoor et al.,2015) suggested a novel ranking algorithm for GFNs based on Euclidean distance, Chutia (2017) developed a modified

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epsilon deviation approach for ranking FNs. De Hierro et al. (2018) proposed a new method to rank FNs and its application to real economic data, Dombi and Jónás (2020) proposed a new ranking algorithm to rank FNs using a probability-based preference intensity index method, Patra (2021) introduced to rank generalized trapezoidal fuzzy numbers (GTrFNs) considering FNs mean position, area, and perimeter as major factors and Hop (2022) proposed a new ranking method using relative relationships and shape characteristics of FNs.

Some of the methods existing in the literature cannot rank FNs with their images, symmetric FNs, and non-normal FNs. To our knowledge, there is no fuzzy ranking method based on volumes, and therefore, in this paper, we present a new ranking method based on the volume of solid obtained by the revolving MF of the GFNs about the axis of support (x-axis). First, we calculate the volume generated by revolving the fuzzy membership function about the axis of support, and then, the FNs score is defined by using the volumes of the left and right positive sides and negative sides of FN, along with the centroid of the FN. This score serves as a defuzzified value of FN and is used to rank FNs. The rest of the paper is structured as follows: Section 2 introduces definitions, Section 3 introduces the novel ranking approach, and Section 4 demonstrates the properties of the score function. Some reasonable properties of the proposed method are presented in Section 5 and Section 6 illustrate the study's numerical examples. The validity of the proposed method is justified in Section 7 through a comparative study with other existing methods in the literature and finally, the conclusions are presented in Section 8.

2. Preliminaries

The definitions of GFNs in this section are drawn from (Zimmermann, 2013).

Definition 2.1 A fuzzy set $\bar{A}$ is a function from universe of discourse X to $[0,1]$, where every element in X has a grade of membership in $[0,1]$.

$$\bar{A} = \{(x, \mu_{\bar{A}}(x)) | x \in X\}$$

where $\mu_{\bar{A}}(x)$ is called the membership function of x in $\bar{A}$.

Definition 2.2 A FN $\bar{A}$ is a fuzzy subset of real line R with MF $f_{\bar{A}}$ satisfying below properties:

1. $f_{\bar{A}}$ is a continuous from R to $[0, h]$,

2. $f_{\bar{A}}$ is strictly increasing on $[k_1, k_2]$,

3. $f_{\bar{A}}(x) = h$, for all $x \in [k_2, k_3]$,

4. $f_{\bar{A}}$ is strictly decreasing on $[k_3, k_4]$,

5. $f_{\bar{A}}(x) = 0$, otherwise

The MF of $f_{\bar{A}}$ can be expressed as:

$$f_{\bar{A}} = \begin{cases} f_{\bar{A}}^L(x); & k_1 \leq x \leq k_2, \\ h; & k_2 \leq x \leq k_3, \\ f_{\bar{A}}^R(x); & k_3 \leq x \leq k_4, \\ 0; & otherwise. \end{cases}$$

where $f_{\bar{A}}^L : [k_1, k_2] \to [0, h]$, and $f_{\bar{A}}^R : [k_3, k_4] \to [0, h]$.

Definition 2.3 A GTrFN $\bar{A} = (k_1, k_2, k_3, k_4; h)$, shown in Fig. 1, is a fuzzy subset of the real line R with MF defined as follows:

$$f_{\bar{A}}(x) = \begin{cases} h\left(\dfrac{x - k_1}{k_2 - k_1}\right) \; if \; k_1 \leq x \leq k_2, \\ h \; if \; k_2 \leq x \leq k_3, \\ h\left(\frac{k_4 - x}{k_4 - k_3}\right) \; if \; k_3 \leq x \leq k_4, \\ 0 \; otherwise. \end{cases} \tag{1}$$

here k_1, k_2, k_3, k_4 are real numbers, and $0 \leq h \leq 1$. If $h = 1$, then $\bar{A}$ is called a normal trapezoidal fuzzy number (TrFN) and if $k_2 = k_3$, then $\bar{A} = (k_1, k_2, k_3; h)$ is called generalized triangular fuzzy number (GTFN).

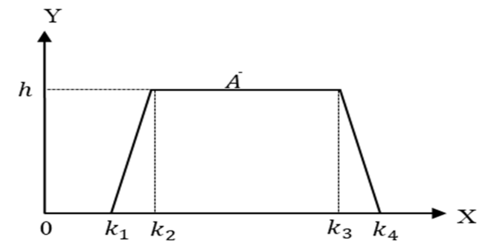


Fig. 1. MF of GTrFN $\bar{A}$.

Fig.2. shows the GTrFN $\bar{A}$ with MF $\mu_{\bar{A}}$, and FN $\bar{S}$ with MF $\mu_{\bar{S}}$ where

$$\mu_{\bar{A}}(x) = \begin{cases} d_1(x); k_1 \le x \le k_2 \\ d_2(x); k_2 \le x \le k_3 \\ d_3(x); k_3 \le x \le k_4 \\ 0; otherwise \end{cases} \tag{2}$$

$$\mu_{\bar{S}}(x) = \begin{cases} s(x) = h; -1 \le x \le 1, \\ 0 \ ; \ otherwise. \end{cases} \tag{3}$$

here $d_1: [k_1, k_2] \to [0,1]$, $d_2: [k_2, k_3] \to [0,1]$, $d_3: [k_3, k_4] \to [0,1]$.

Definition 2.4 (Grewal, 2017) The volume of the solid obtained by revolving an area bounded by $y = f(x)$ between $[a, b]$ about the x-axis is

$$V = \pi \int_a^b y^2 \, dx \tag{4}$$

Definition 2.5 The volume of a solid acquired by revolving the left MF of the GTrFN $\bar{A} = (k_1, k_2, k_3, k_4; h)$, given by Eq. (1), about x-axis is:

$$v_1 = \frac{\pi}{3} h^2 (k_2 - k_1) \tag{5}$$

Definition 2.6 The volume of a solid acquired by rotating the core of the GTrFN $\bar{A} = (k_1, k_2, k_3, k_4; h)$, given by Eq. (1), about x-axis is:

$$v_2 = \frac{\pi}{3} h^2 (k_3 - k_2) \tag{6}$$

Definition 2.7 The volume of a solid acquired by rotating the right MF of the GTrFN $\bar{A} = (k_1, k_2, k_3, k_4; h)$, given by Eq. (1), about x-axis is:

$$v_3 = \frac{\pi}{3} h^2 (k_4 - k_3) \tag{7}$$

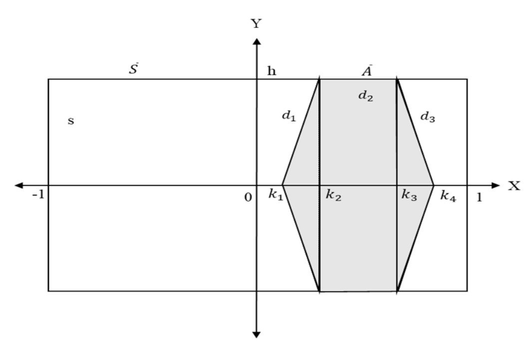


Fig. 2. TrFNs $\bar{A}$ and $\bar{S}$ membership function.

3. New approach for ranking GFNs using volumes

This section introduces a new ranking algorithm for GFNs based on volumes. The suggested technique computes the volumes of the positive and negative sides, as well as the centroid of the GFNs, to estimate the score of each GFN. This score is the defuzzified value of the given GFN and used for ranking purpose. Suppose there are 'n' GFNs $\overline{A_1},\ \overline{A_2}, \overline{A_3}, \ldots, \overline{A_n}$

to be ranked, where $\overline{A_\iota} = (k_{i1}, k_{i2}, k_{i3}, k_{i4}; h_i)$, $-\infty \le k_{i1} \le k_{i2} \le k_{i3} \le k_{i4} \le h_i \le \infty$; $h_i \in [0,1]$, h_i denotes the height of the FN the proposed ranking approach for GFNs is as follows:

Step 1: Calculation of the volumes obtained by revolving the MF about x-axis for each GFN. For this, we do the following:

We divide the volume of GFN $\overline{A_\iota}$ to Left negative volume LN_i, Right negative volume RN_i, Left positive volume LP_i and Right positive volume RP_i, where $1 \le i \le n$, as mentioned below:

i) Left negative volume LN_i : It indicates the volume from GFN $(-1,-1,-1,-1; h)$ to the MF curve of d_1 shown in fig 3(a).

$$LN_i = \int_{-1}^{k_2} [s(x)]^2 dx - \int_{k_1}^{k_2} [d_1(x)]^2 dx$$
$$\Rightarrow LN_i = \frac{\pi}{3} h^2 (2k_2 + k_1 + 3) \tag{8}$$

ii) Right negative volume RN_i: It indicates the volume from the GFN $(-1,-1,-1,-1; h)$ shown in fig 3(b) to the MF curve of d_3.

$$RN_i = \int_{-1}^{k_3} [s(x)]^2 dx - \int_{k_3}^{k_4} [d_3(x)]^2 dx$$
$$\Rightarrow RN_i = \frac{\pi}{3} h^2 (2k_3 + k_4 + 3) \tag{9}$$

iii) Left positive volume LP_i: It indicates the volume from the MF curve of d_1 to the GFN $(1,1,1,1; h)$ shown in fig 3(c).

$$LP_i = \int_{k_2}^{1} [s(x)]^2 dx - \int_{k_1}^{k_2} [d_1(x)]^2 \, dx$$
$$\Rightarrow LP_i = \frac{\pi}{3} h^2 (3 - 2k_2 - k_1) \tag{10}$$

iv) Right positive volume RP_i : The volume from the MF curve of d_3 to the GFN $(1,1,1,1; h)$ illustrated in fig 3(d).

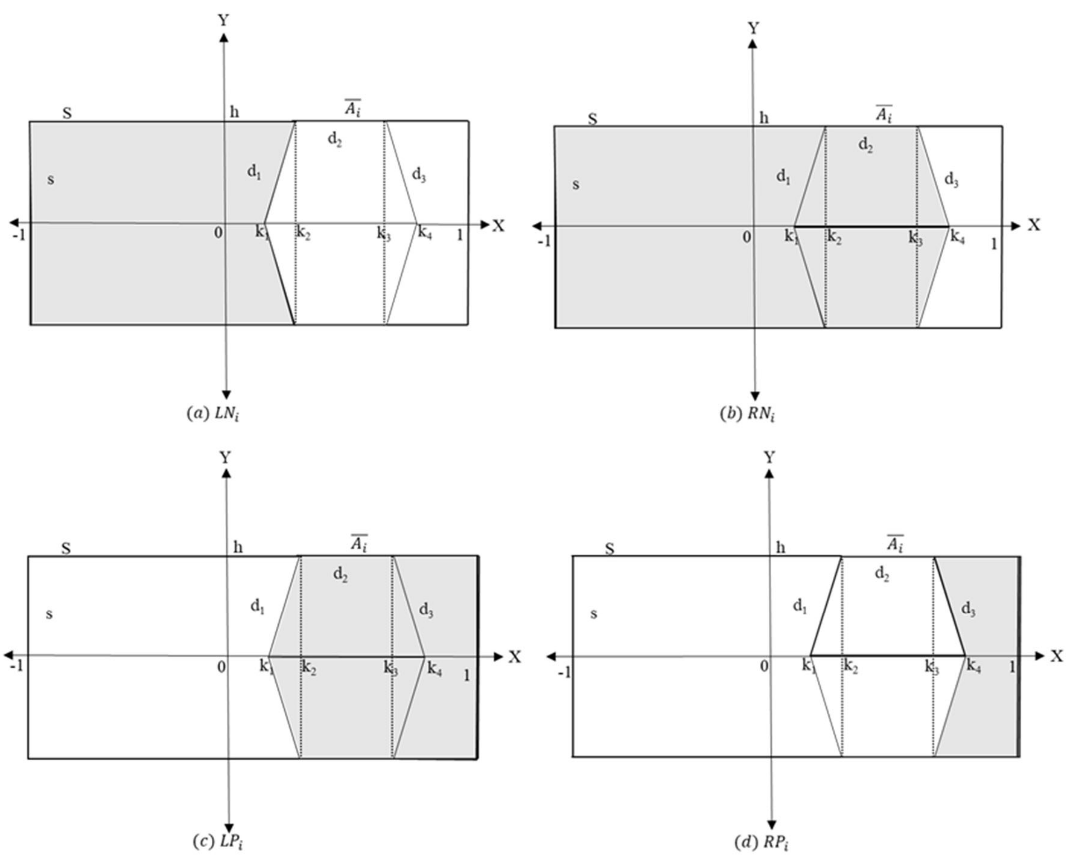


Fig. 3. The negative and positive volumes of GFN $\overline{A_\iota}$.

$$RP_i = \int_{k_3}^{1} [s(x)]^2 dx - \int_{k_3}^{k_4} [d_3(x)]^2 dx$$

$$\Rightarrow RP_i = \frac{\pi}{3} h^2 (3 - 2k_3 - k_3) \tag{11}$$

Step 2: Compute the sums M_i and N_i of positive and negative sides volumes of the GFN $\bar{A_i}$, where

$$M_i = LN_i + RN_i \tag{12}$$

$$N_i = LP_i + RP_i \tag{13}$$

where i is in $[1, n]$.

Step 3: Compute centroid $C(\bar{A_i})$ for every GFN $\bar{A_i}$ given below:

$$C(\bar{A_i}) = \frac{1}{3}[k_1 + k_2 + k_3 + k_4 - \frac{k_4 k_3 - k_1 k_2}{(k_4 + k_3) - (k_1 + k_2)}] \tag{14}$$

Note: For crisp number $\bar{A_i} = (k, k, k, k; h)$ we use the following centroid formula i.e.

$$C(\bar{A_i}) = k \tag{15}$$

Step 4: The ranking score$(\bar{A_i})$ for each GFN $\bar{A_i}$ is defined as below:

$$score(\bar{A_i}) = \frac{\bar{M_i} - \bar{N_i}}{\bar{M_i} + \bar{N_i} + (1 - |C(\bar{A_i})|)} \tag{16}$$

here $\bar{M_i} - \bar{N_i} = \frac{2\pi}{3} h^2 (k_1 + 2k_2 + 2k_3 + k_4)$ and $\bar{M_i} + \bar{N_i} = 4\pi h^2$

Therefore,

$$score(\bar{A_i}) = \frac{\frac{2\pi}{3} h^2 (k_1 + 2k_2 + 2k_3 + k_4)}{4\pi h^2 + 1 - |\frac{1}{3}[k_1 + k_2 + k_3 + k_4 - \frac{k_4 k_3 - k_1 k_2}{(k_4 + k_3) - (k_1 + k_2)}]|} \tag{17}$$

where i is in $[1, n]$.

Ranking procedure: If $\bar{A}_1$ and $\bar{A}_2$ are two FNs, using scores given by Eq. (17), we define the following ranking order:

i) If $score(\bar{A}_1) < score(\bar{A}_2)$ then $\bar{A}_1$ is less preferred to $\bar{A}_2$, expressed as $\bar{A}_1 \prec \bar{A}_2$.

ii) If $score(\bar{A}_1) > score(\bar{A}_2)$ then $\bar{A}_1$ is more preferred to $\bar{A}_2$, expressed as $\bar{A}_1 \succ \bar{A}_2$.

iii) If $score(\bar{A}_1) = score(\bar{A}_2)$ then $\bar{A}_1$ is equal to $\bar{A}_2$, expressed as $\bar{A}_1 \approx \bar{A}_2$.

4. Properties

In this section, some properties of the new ranking technique are presented.

Property 4.1 Suppose $\overline{A_1} = (k_1, k_2, k_3, k_4; h)$ is a GFN and

$k_1 + k_2 + k_3 + k_4 = 0$, and $-1 \le k_1 \le k_2 \le k_3 \le k_4 \le 1$, then $score(\overline{A_1}) = 0$.

Proof: To satisfy the equation $k_1 + k_2 + k_3 + k_4 = 0$, we put $k_3 = -k_2$ *and* $k_4 = -k_1$, shown in Fig. 4.

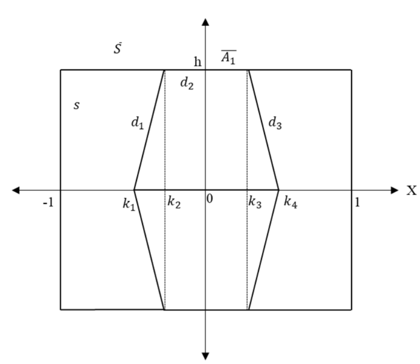


Fig. 4. GFN $\overline{A_1}$ and S'.

By replacing the values $k_3 = -k_2$ *and* $k_4 = -k_1$ in Eq. (17), we get

$$\bar{M}_1 - \bar{N}_1 = \frac{2\pi}{3} h^2 (k_1 + 2k_2 - 2k_2 - k_1) = 0$$

$$C(\overline{A_1}) = \frac{1}{3}\left[k_1 + k_2 - k_2 - k_1 - \frac{k_1 k_2 - k_1 k_2}{(-k_1 - k_2) - (k_1 + k_2)}\right] = 0$$

$$score(\overline{A_1}) = \frac{\bar{M}_1 - \bar{N}_1}{\bar{M}_1 + \bar{N}_1 + (1 - |0|))} = \frac{0}{4\pi h^2 + 1} = 0$$

Property 4.2 If $\overline{A_1} = (k_1, k_2, k_3, k_4; h)$ is a GFN and $\overline{A_2} = (-k_4, -k_3, -k_2, -k_1; h)$, is the image of $\overline{A_1}$ where $-1 \le k_1 \le k_2 \le k_3 \le k_4 \le 1$, then $score(\overline{A_1}) = -score(\overline{A_2})$.

Proof: Given $\overline{A_1} = (k_1, k_2, k_3, k_4, h)$, where $-1 \le k_1 \le k_2 \le k_3 \le k_4 \le 1$ shown in Fig.5. and from Eq. (17)

$$score(\overline{A_1}) = \frac{\frac{2\pi}{3} h^2 (k_1 + 2k_2 + 2k_3 + k_4)}{4\pi h^2 + \left[1 - \frac{1}{3}\left[k_1 + k_2 + k_3 + k_4 - \frac{k_4 k_3 - k_1 k_2}{(k_4 + k_3) - (k_1 + k_2)}\right]\right]}$$

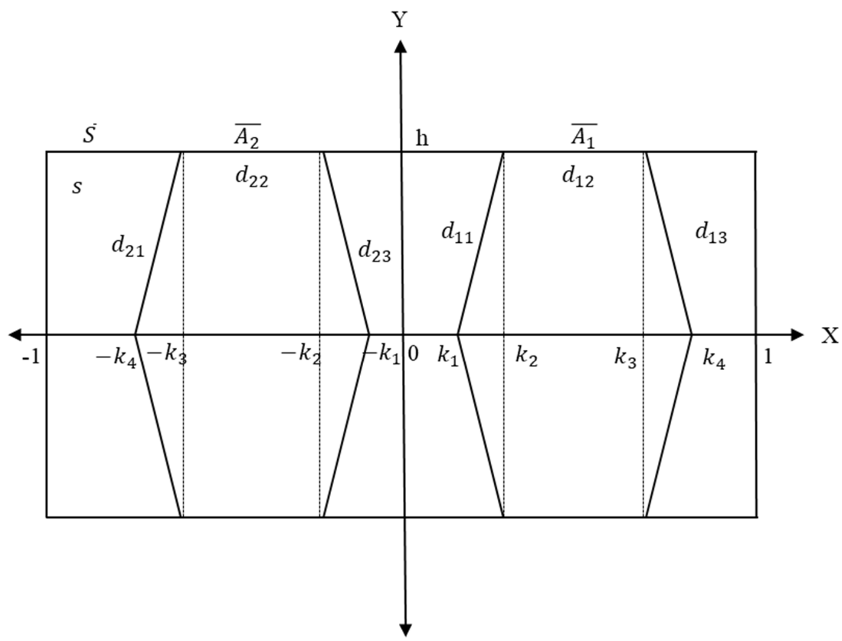


Fig. 5. GFNs $\overline{A_1} = (k_1, k_2, k_3, k_4; h), \overline{A_2} = (-k_4, -k_3, -k_2, -k_1; h)$ and $\bar{S}$.

Given $\overline{A_2} = (-k_4, -k_3, -k_2, -k_1, h)$, where $-1 \le k_1 \le k_2 \le k_3 \le k_4 \le 1$, and from Eq. (17), we

$$score(\overline{A_2}) = \frac{-(\frac{2\pi}{3} h^2 (k_1 + 2k_2 + 2k_3 + k_4))}{4\pi h^2 + \left[1 - \frac{1}{3}\left[k_1 + k_2 + k_3 + k_4 - \frac{k_4 k_3 - k_1 k_2}{(k_4 + k_3) - (k_1 + k_2)}\right]\right]}$$

$$\therefore score(\overline{A_1}) = -score(\overline{A_2})$$

Property 4.3 If $\overline{A_1} = (1,1,1,1; h)$, then $score(\overline{A_1}) = 1$.

Proof: Given $\overline{A_1} = (1,1,1,1; h)$, shown in Fig.6. from Eq. (15) we have $C(\overline{A_1}) = 1$

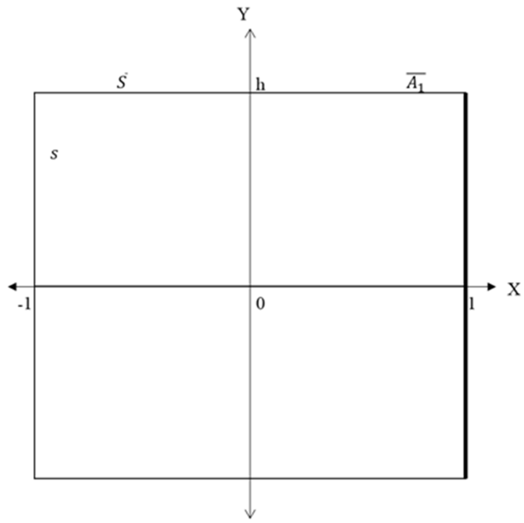


Fig. 6. GFN $\overline{A_1} = (1,1,1,1;h)$ and $\dot{S}$.

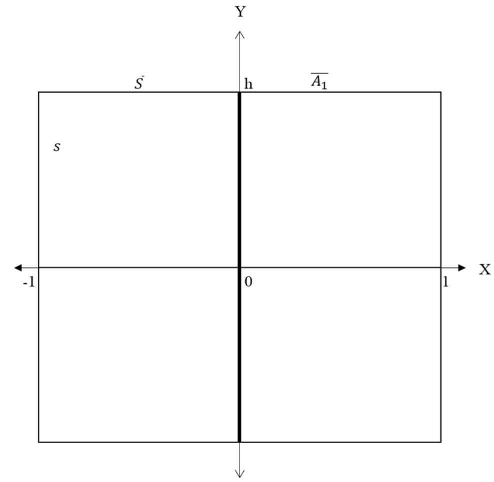


Fig. 7. GFN $\overline{A_1} = (0,0,0,0;h)$ and $\dot{S}$.

From Eq.(17), $\bar{M}_1 - \bar{N}_1 = \frac{2\pi}{3}h^2(1 + 2(1) + 2(1) + 1) = \frac{2\pi}{3}h^2(6) = 4\pi h^2$ and

$$score(\overline{A_1}) = \frac{4\pi h^2}{4\pi h^2 + [1 - 1]} = 1$$

Property 4.4 If $\overline{A_1} = (0,0,0,0;h)$, then $score(\overline{A_1}) = 0$.

Proof: Given $\overline{A_1} = (0,0,0,0;h)$,shown in Fig.7. from Eq. (15) we have $C(\overline{A_1}) = 0$

From Eq.(17), $\bar{M}_1 - \bar{N}_1 = \frac{2\pi}{3}h^2(0 + 2(0) + 2(0) + 0) = \frac{2\pi}{3}h^2(0) = 0$ and

$$score(\overline{A_1}) = \frac{0}{4\pi h^2 + [1 - 0]} = 0$$

Property 4.5 If $\overline{A_1} = (-1,-1,-1,-1;h)$, then $score(\overline{A_1}) = -1$.

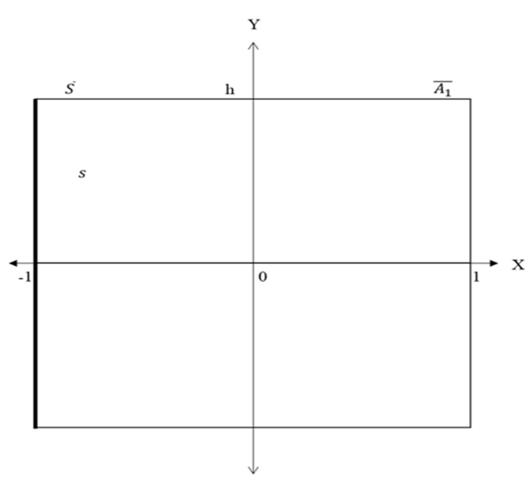


Fig. 8. GFN $\overline{A_1} = (-1,-1,-1,-1;h)$ and $\dot{S}$.

Proof: Given $\overline{A_1} = (-1,-1,-1,-1;h)$, shown in Fig.8. from Eq. (15) we have $C(\overline{A_1}) = -1$

From Eq.(17), $\bar{M}_1 - \bar{N}_1 = \frac{2\pi}{3}h^2((-1) + 2(-1) + 2(-1) + (-1)) = \frac{2\pi}{3}h^2(-6) = -4\pi h^2$ and

$$score(\overline{A_1}) = \frac{-4\pi h^2}{4\pi h^2} = -1$$

5. Reasonable properties

In this section, we present some reasonable properties the proposed method (Wang & Kerre, 2001)

Let M be the ordering approach, and F be the set of fuzzy quantities for which the method M can be applied. **A** is a finite subset of F and $\bar{A}$ and $\bar{B}$ are two elements in **A**.

Theorem 5.1 Let **A** be a finite subset of F and $\bar{A} \in \boldsymbol{A}$, $\bar{A} \succ \bar{A}$ by M on **A**.

Proof: For any arbitrary FN $\bar{A} \in \boldsymbol{A}$, $score(\bar{A})$ is a real value, say v

In real sets $v \geq v$. Hence, we have $\bar{A} \succcurlyeq \bar{A}$.

Theorem 5.2 Let **A** be a finite subset of F and $(\bar{A}, \bar{B}) \in \boldsymbol{A}^2$, $\bar{A} \succcurlyeq \bar{B}$ and $\bar{B} \succcurlyeq \bar{A}$, by M on **A**, then $\bar{A} \sim \bar{B}$ by M on **A**.

Proof: Consider $(\bar{A}, \bar{B}) \in \mathbf{A}^2$ with $\bar{A} \succcurlyeq \bar{B}$, $\bar{B} \succcurlyeq \bar{A}$.

Let $score(\bar{A}) = p$ and $score(\bar{B}) = q$;

Now, $\bar{A} \succcurlyeq \bar{B}$

$\Rightarrow score(\bar{A}) \geq score(\bar{B})$

$\Rightarrow p \geq q$.

Now, $\bar{B} \succcurlyeq \bar{A}$

$\Rightarrow score(\bar{B}) \geq score(\bar{A})$

$\Rightarrow q \geq p$.

The above two inequalities are positive on real numbers only if $p = q$.
Hence, $\bar{A} \sim \bar{B}$.

Theorem 5.3 Let **A** be a finite subset of F and $(\bar{A}, \bar{B}, \bar{C}) \in \boldsymbol{A}^3$, $\bar{A} \succcurlyeq \bar{B}$ and $\bar{B} \succcurlyeq \bar{C}$ by M on **A**, then $\bar{A} \succcurlyeq \bar{C}$ by M on **A**.

Proof: Consider three FNs $(\bar{A}, \bar{B}, \bar{C}) \in \boldsymbol{A}^3$ with $\bar{A} \succcurlyeq \bar{B}$ and $\bar{B} \succcurlyeq \bar{C}$ by M on **A**.

Let $score(\bar{A}) = F_{\bar{A}}, score(\bar{B}) = F_{\bar{B}}, score(\bar{C}) = F_{\bar{C}}$

We know that $\bar{A} \succcurlyeq \bar{B}$ and $\bar{B} \succcurlyeq \bar{C}$ then

$\Rightarrow score(\bar{A}) \geq score(\bar{B})$; $score(\bar{B}) \geq score(\bar{C})$

$\Rightarrow F_{\bar{A}} \geq F_{\bar{B}}$; $F_{\bar{B}} \geq F_{\bar{C}}$

$\Rightarrow F_{\bar{A}} \geq F_{\bar{C}}$

$\Rightarrow \bar{A} \succcurlyeq \bar{C}$.

Theorem 5.4 Let **A** be a finite subset of F and $(\bar{A}, \bar{B}) \in \boldsymbol{A}^2$,

(i) if $\inf Supp(\bar{A}) > \sup Supp(\bar{B})$, then $\bar{A} \succcurlyeq \bar{B}$ by M on **A**.

(ii) if $\inf Supp(\bar{A}) > \sup Supp(\bar{B})$, then $\bar{A} \succ \bar{B}$ by M on **A**.(stronger version of (i))

Proof: Since (ii) is stronger than (i), (ii) is proved.

Let **A** be a finite subset of F and $(\bar{A}, \bar{B}) \in \boldsymbol{A}^2$ with $\inf Supp(\bar{A}) > \sup Supp(\bar{B})$.

Clearly, $score(\bar{A}) \geq \inf Supp(\bar{A})$

And $score(\bar{B}) \leq \sup Supp(\bar{B})$

Therefore, $score(\bar{A}) \geq \inf Supp(\bar{A}) > \sup Supp(\bar{B}) \geq score(B)$.

Hence $\bar{A} \succ \bar{B}$.

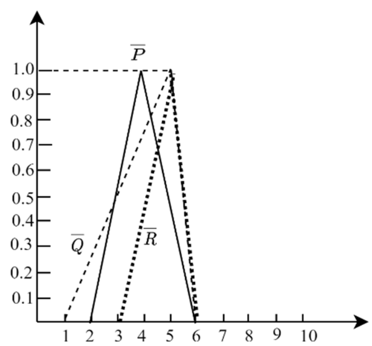
Theorem 5.5 If F and F' are two arbitrary finite sets of fuzzy quantities in which M can be applied and $\bar{A}$ and $\bar{B}$ are in $F \cap F'$.Then $\bar{A} \succ \bar{B}$ by M on F.

Proof: The final ranking order of $\bar{A}$ and $\bar{B}$ is solely dependent on the score values of $\bar{A}$ and $\bar{B}$ and has nothing to do with any other fuzzy quantities in F or F'. Hence, the ranking of $\bar{A}$ and $\bar{B}$ on F and F' is same that of the final ranking order.

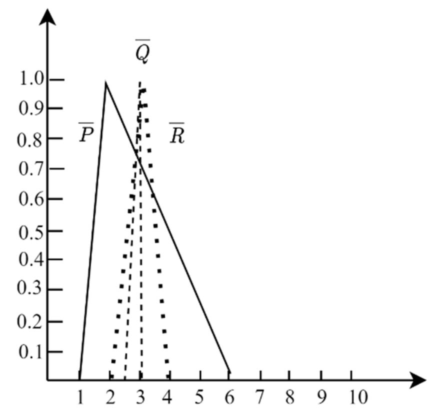
6. Numerical Examples

In the part that follows, we will use numerical examples to demonstrate the ranking process of the suggested approach.

Example 6.1 Let $\bar{P} = (2,4,6), \bar{Q} = (1,5,6), \bar{R} = (3,5,6)$ be three fuzzy numbers taken from (Yu et al., 2013) shown in Fig. 9



$\overline{P} = (2,4,6)$
$\overline{Q} = (1,5,6)$
$\overline{R} = (3,5,6)$
Fig.9.



$\overline{P} = (1,2,6)$
$\overline{Q} = (2.5, 2.75, 3)$
$\overline{R} = (2,3,4)$
Fig.10.

By applying the proposed method, we get $score(\bar{P}) = 5.2537, score(\bar{Q}) = 5.9104$ and $score(\bar{R}) = 6.8235$, so the ranking order is $\bar{P} \prec \bar{Q} \prec \bar{R}$. It is to be noted that (Wang et al., 2009) failed to rank the FNs, this is also pointed out by (Nejad and Mashinchi, 2011).

Example 6.2 Consider three FNs $\bar{P} = (1,2,6), \bar{Q} = (2.5,2.75,3), \bar{R} = (2,3,4)$

taken from (Yu et al., 2013) shown in Fig.10

By applying the proposed method, we get $score(\bar{P}) = 2.9729, score(\bar{Q}) = 3.19472, score(\bar{R}) = 3.56757$ so the ranking order is $\bar{P} \prec \bar{Q} \prec \bar{R}$.

Also, $score(-\bar{P}) = -2.9729, score(-\bar{Q}) = -3.19472, score(-\bar{R}) = -3.56757$

This shows that if $\bar{P} \prec \bar{Q} \prec \bar{R}$ then $-\bar{R} \prec -\bar{Q} \prec -\bar{P}$.

It is to be noted that (Wang et al.,2009) failed to give correct ranking ordering of images.

Example 6.3 Consider the following two fuzzy sets taken from (Yu et al., 2013) shown in Fig. 11

$$g_{\bar{P}}(x) = \begin{cases} x-1, & if\ x \in [1,2] \\ 3-x, & if\ x \in [2,3] \\ 0, & otherwise. \end{cases}$$

$$g_{\bar{Q}}(x) = \begin{cases} [1-(x-2)^2]^{\frac{1}{2}}, & if\ x \in [1,2] \\ [1-\frac{1}{4}(x-2)^2]^{\frac{1}{2}}, & if\ x \in [2,4] \\ 0, & otherwise. \end{cases}$$

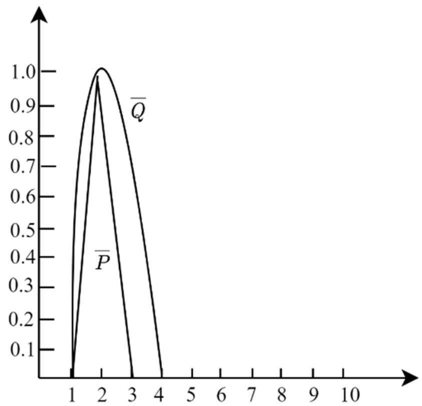
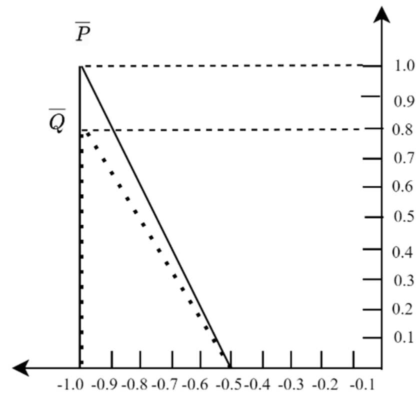


Fig.11



$\bar{P} = (-.1.0, -1.0, -1.0, -0.5; 1)$
$\bar{Q} = (-1.0, -1.0, -1.0, -0.5; 0.8)$
Fig.12.

By applying the proposed method, we get $score(\bar{P}) = 2.1730$ and $score(\bar{Q}) = 6.338$.Therefore, the ranking order is $\bar{P} \prec \bar{Q}$.

It is to be observed that (Wang et al., 2009) and (Nejad and Mashinchi, 2011) failed to give correct ordering of images.

Example 6.4 Consider the following two GFN sets taken from (Jiang et al., 2015) shown in Fig. 12

$\bar{P} = (-1.0, -1.0, -1.0, -0.5; 1), \bar{Q} = (-1.0, -1.0, -1.0, -0.5; 0.8)$

By applying the proposed method, we get $score(\bar{P}) = -0.9047$ and $score(\bar{Q}) = -0.5790$. Therefore, the ranking order is $\bar{P} \prec \bar{Q}$. It is to be noted that (Chen et al. 2012) method failed to rank the FNs.

7. New approach versus existing techniques-comparative study

The GFN sets obtained from (Chen et al, 2012, Chen and Chen, 2009, Deng and Liu, 2005) shown in Fig 13 and Fig.14 are used for the purpose of comparative study.

We compare the suggested ranking method's ranking outcomes to several existing approaches such as (Cheng, 1998), (Yager, 1978), (Chu and Tsao's, 2002), (Murakami et al., 1983), (Chen and Sanguansat, 2011), (Chen and Chen, 2007), (Chen and Chen, 2009) and (Chen et al., 2012).

7.1 Comparative study I

Consider different sets of FNs shown in Fig. 13 and Table 1 displays the comparative results. We observe from Table 1 the following:

1) For Set-1 with FNs $\bar{U} = (0.1,0.3,0.3,0,5; 1), \bar{V} = (0.3,0.5,0.5,0.7; 1)$, the proposed method gives the same results as all other methods. i.e. $\bar{U} \prec \bar{V}$.

2) For Set-2 with FNs $\bar{U} = (0.1,0.2,0.4,0.5; 1), \bar{V}$= (0.1,0.3,0.3,0.5;1), the proposed method results are same as all other methods where the ranking sequence is $\bar{U} = \bar{V}$, except (Murakami et al., 1983; Chen & Chen, 2007; Chen & Chen, 2009) methods has ranking order $\bar{V} \succ \bar{U}$.

3) For Set-3 with FNs $\bar{U} = (0.1,0.3,0.3,0.5; 1), \bar{V} = (0.2,0.3,0.3,0.4; 1)$, the proposed method results are same as all other methods i.e., $\bar{U} = \bar{V}$, but (Chen & Chen, 2007; Chen & Chen, 2009) has ranking order $\bar{V} \succ \bar{U}$.

4) For Set-4 with FNs $\bar{U} = (0.1,0.3,0.3; 0.5; 0.8), \bar{V} = (0.1,0.3,0.3,0.5; 1)$, the proposed method results are same as all other methods which is $\bar{V} \succ \bar{U}$,but (Yager,1978) ranking sequence is $\bar{U} = \bar{V}$.

5) For Set-5 with FNs $\bar{U} = (0.1,0.2,0.4,0.5; 1)$, $\bar{V} = (1.0,1.0,1.0,1.0; 1)$, the proposed method results match with (Chen & Chen, 2009; Chen & Sanguansat's, 2011; Chen et al., 2012). These approaches can compute the scores and determine the ranking sequence as $\bar{V} \succ \bar{U}$, but (Cheng, 1998; Chu & Tsao, 2002; Murakami et al., 1983; Yager, 1978) failed to give the calculated value of the crisp FN $\bar{V}$.

6) For Set-6 with FNs $\bar{U} = (-0.5, -0.3, -0.3, -0.1; 1)$, $\bar{V} = (0.1, 0.3, 0.3, 0.5; 1)$, the proposed method results are same as all other methods that is $\bar{V} \succ \bar{U}$, as $\bar{V}$ is image of $\bar{U}$ (Cheng,1998) ranking order $\bar{U} = \bar{V}$ does not coincide with human intuition.

7) For Set-7 with FNs $\bar{U} = (0.3, 0.5, 0.5, 1.0; 1), \bar{V} = (0.1, 0.6, 0.6, 0.8; 1)$, the proposed method results are same as all other methods which is $\bar{U} \succ \bar{V}$.

8)For Set-8 with FNs $\bar{U} = (0, 0.4, 0.6, 0.8; 1), \bar{V} = (0.2, 0.5, 0.5, 0.9; 1)$ and $\bar{W} = (0.1, 0.6, 0.7, 0.8; 1)$,the proposed method results are same as all other methods $\bar{W} \succ \bar{V} \succ \bar{U}$ except (Murakami et al., 1983; Yager, 1978; Chen & Chen, 2007) which has the ranking order is $\bar{V} \succ \bar{W} \succ \bar{U}$.

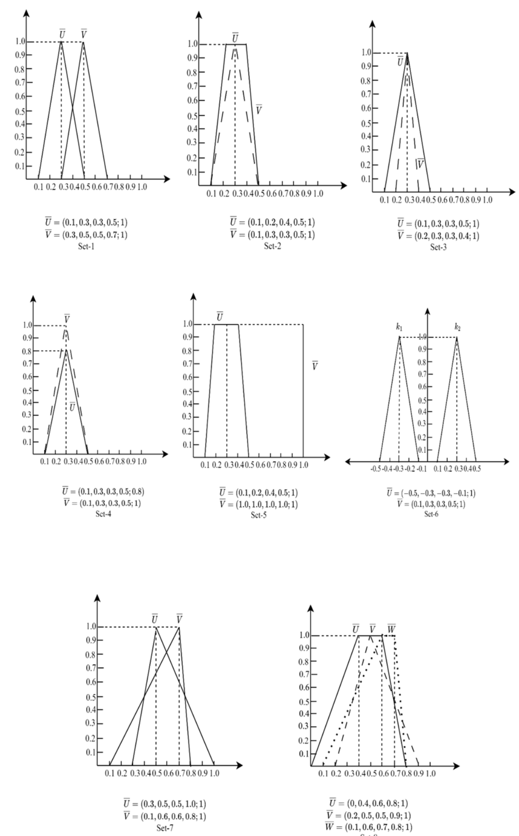


Fig. 13. GFNs sets

Table 1
A comparison of the suggested method's outcomes with existing techniques

Ranking Approaches	Set-1		Set-2		Set-3		Set-4	
	$\bar{U}$	$\bar{V}$	$\bar{U}$	$\bar{V}$	$\bar{U}$	$\bar{V}$	$\bar{U}$	$\bar{V}$
Cheng, (1998)	0.583	0.707	0.583	0.583	0.583	0.583	0.461	0.583
Chu and Tsao, (2002)	0.15	0.25	0.15	0.15	0.15	0.15	0.12	0.15
Murakami et al., (1983)	0.3	0.5	0.3	0.416	0.3	0.3	0.23	0.3
Yager, (1978)	0.3	0.5	0.3	0.3	0.3	0.3	0.3	0.3
Chen and Chen, (2007)	0.445	0.488	0.423	0.445	0.445	0.472	0.356	0.445
Chen and Chen, (2009)	0.257	0.429	0.253	0.257	0.257	0.277	0.206	0.257
Chen and Sanguansat, (2011)	0.3	0.5	0.3	0.3	0.3	0.3	0.282	0.3
Chen et al., (2012)	0.25	0.4	0.25	0.25	0.25	0.25	0.246	0.25
Suggested Approach	0.284	0.464	0.284	0.284	0.284	0.284	0.276	0.284

Ranking Approaches	Set-5		Set-6		Set-7		Set-8		
	$\overline{U}$	$\overline{V}$	$\overline{U}$	$\overline{V}$	$\overline{U}$	$\overline{V}$	$\overline{U}$	$\overline{V}$	$\overline{W}$
Cheng, (1998)	0.424	N/A	0.583	0.583	0.767	0.724	0.68	0.725	0.746
Chu and Tsao, (2002)	0.15	N/A	-0.15	0.15	0.287	0.261	0.228	0.262	0.278
Murakami et al., (1983)	0.416	N/A	-0.3	0.3	0.6	0.5	0.44	0.53	0.525
Yager, (1978)	0.3	N/A	-0.3	0.3	0.6	0.5	0.44	0.53	0.525
Chen and Chen, (2007)	0.424	0.86	0.445	0.747	0.412	0.4	0.371	0.415	0.397
Chen and Chen, (2009)	0.253	1.0	-0.257	0.257	0.442	0.404	0.335	0.407	0.419
Chen and Sanguansat, (2011)	0.3	1.0	-0.3	0.3	0.575	0.525	0.45	0.525	0.55
Chen et al., (2012)	0.255	1.0	-0.255	0.255	0.51	0.477	0.4	0.466	0.505
Suggested Approach	0.284	1.0	-0.284	0.284	0.533	0.528	0.446	0.498	0.583

Note: "N/A" means the method cannot compute the ranking value.

7.2 Comparative study II

In the following, we study the performance of the suggested approach with (Cheng, 1998; Yager, 1978; Murakami et al., 1983; Deng and Liu, 2005; Chu and Tsao, 2002; Chen and Chen, 2007; Chen and Chen, 2009; Chen et al., 2012; Chen and Sanguansat, 2011) for the three GFNs shown in Fig. 14 taken from (Deng and Liu, 2005, Chen and Chen, 2009). Table 2 displays the results. From Table 2, we can see that

1) For Set-1 with $\overline{U} = (0.1,0.3,0.3,0.5;1), \overline{V} = (0.2,0.3,0.3,0.4;1), \overline{W} = (1.0,1.0,1.0,1.0;1)$ as FNs the proposed method results match with (Chen et al., 2012), (Chen & Sanguansat, 2011), i.e.,

$\overline{W} \succ \overline{U} \approx \overline{V}$. However, (Yager, 1978), (Murakami et al., 1983) and (Cheng, 1998) failed to give the calculated value of the crisp FN $\overline{W}$ and their ranking order is $\overline{U} = \overline{V}$. Also, (Chen & Chen, 2009; Chen & Chen, 2007; Deng & Liu, 2005) can compute the ranking score of crisp-value FN $\overline{W}$, however the order of (Deng & Liu, 2005; Chen & Chen, 2007; Chen & Chen, 2009) is

$\overline{W} \succ \overline{V} \succ \overline{U}$.

2) For Set-2 with $\overline{U} = (0.1,0.3,0.3,0.5;1)$ *and* $\overline{V} = (-0.5,-0.3,-0.3,-0.1;1)$, the proposed method ranking order match with (Chen et al., 2012; Chen & Sanguansat, 2011; Chen & Chen, 2009; Chen & Chen, 2007; Deng & Liu,2005; Murakami et al.,1983; Yager,1978) which is $\overline{U} \succ \overline{V}$. But (Cheng ,1998) ranking order $\overline{U} \approx \overline{V}$ does not coincide with human intuition as $\overline{V}$ is image of $\overline{U}$.

3) For Set-3 with $\overline{U} = (0.01,0.01,0.01,0.01;0.8)$ *and* $\overline{V} = (-0.01,-0.01,-0.01,-0.01;1)$, the proposed method results match with (Chen et al., 2012; Cheng & Chen, 2009; Chen & Sanguansat, 2011; Deng & Liu, 2005) which is $\overline{U} \succ \overline{V}$.

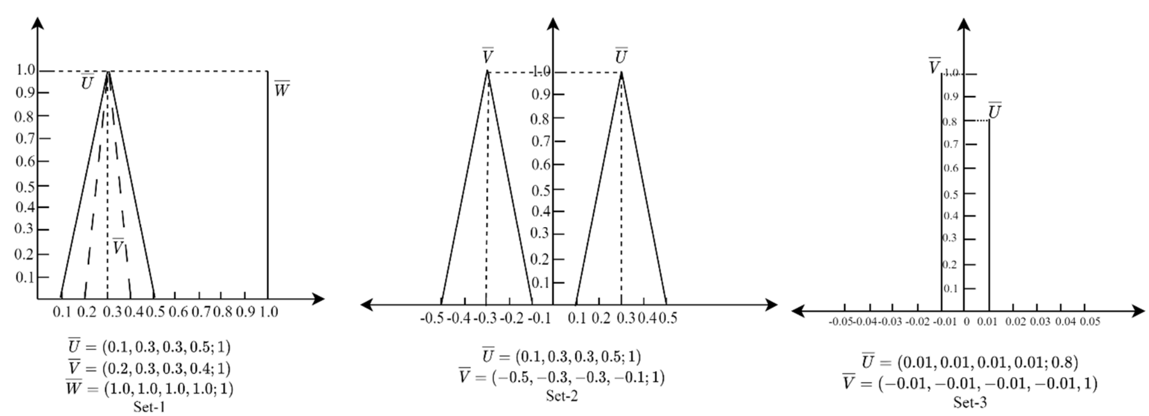


Fig. 14. Sets of GTrFNs

Table 2
A comparison of the suggested method's outcomes with existing techniques

Ranking Approaches	Set-1			Set-2		Set-3	
	$\bar{U}$	$\bar{V}$	$\bar{W}$	$\bar{U}$	$\bar{V}$	$\bar{U}$	$\bar{V}$
Yager, (1978)	0.3	0.3	N/A	0.3	-0.3	N/A	N/A
Murakami et al., (1983)	0.3	0.3	N/A	0.3	-0.3	N/A	N/A
Cheng, (1998)	0.583	0.583	N/A	0.583	0.583	N/A	N/A
Deng and Liu, (2005)	0.621	0.624	1.0	0.621	0.375	0.505	0.495
Chen and Chen, (2007)	0.445	0.472	0.860	0.747	0.445	0.4	0.5
Chen and Chen, (2009)	0.257	0.277	1.0	0.257	-0.257	0.008	-0.01
Chen and Sanguansat, (2011)	0.3	0.3	1.0	0.3	-0.3	0.009	-0.001
Chen et al.,(2012)	0.255	0.255	1.0	0.255	-0.255	0.007	-0.008
Suggested Approach	0.284	0.284	1.0	0.284	-0.284	0.008	-0.009

Note: "N/A" means the method cannot compute the ranking value.

8. Conclusion

In the present study, we developed a novel ranking approach for ranking GFNs based on volume generated by the revolution of membership function about x-axis. We estimated the ranking scores of GFNs by assessing the volumes of the positive and negative sides, as well as the centroid of the GFNs. This ranking score is the defuzzified value of the FN and used for ordering of FNs. The suggested approach overcome some of the weaknesses in existing methods and can effectively rank different types of FNs, along with their images and crisp numbers. A comparative study is conducted to study and compare the findings of the proposed approach with those of other existing ranking algorithms and found that this new ranking approach is doing reasonably well. The proposed approach is useful in fuzzy risk assessment, decision making, and other fuzzy application systems.

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