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A novel multi-criteria group decision-making approach using aggregation operators and weight determination method for supplier selection problem in hesitant Pythagorean fuzzy environment

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A B S T R A C T

Uncertainty is an important factor in the decision-making process. Hesitant Pythagorean fuzzy sets (HPFS), a combination of Pythagorean and hesitant fuzzy sets, proved as a significant tool to handle uncertainty. Well-defined operational laws and attribute weights play an important role in decision-making. Thus, the paper aims to develop new Trigonometric Operational Laws, a weight determination method, and a novel score function for group decision-making (GDM) problems in the HPF environment. The approach is presented in three phases. The first phase defines new operational laws with sine trigonometric function incorporating its special properties like periodicity, symmetricity, and restricted range hence compared with previously defined aggregation operators they are more likely to satisfy the decision maker preferences. Properties of trigonometric operational laws (TOL) are studied and various aggregation operators are defined. To measure the relationship between arguments, the operators are combined with the Generalized Heronian Mean operator. The flexibility of operators is increased by the use of a real parameter λ to express the risk preference of experts. The second phase defines a novel weight determination method, which separately considers the membership and non-membership degrees hence reducing the information loss and the third phase conquers the shortcomings of previously defined score functions by defining a novel score function in HPFS. To further increase the flexibility of defined operators they are extended in the environment with unknown or incomplete attribute weights. The effectiveness of the GDM model is verified with the help of a supplier selection problem. A detailed comparative analysis demonstrates the superiority, and sensitivity analysis verifies the stability of the proposed approach.

1. Introduction

The fuzzy set theory came into existence by the great efforts of (Zadeh, 1965) to manage uncertainties that remain in the real world. Then there were some extensions of the fuzzy sets. The Intuitionistic fuzzy set (IFS) (Atanassov 1986) is one of the extensions of fuzzy sets characterized by a degree of membership and non-membership between 0 and 1, so it deals with uncertainty very accurately. IFS limits the sum of membership and non-membership to 1. Interval-valued IFS (IVIFS) (Atanassov 1989) is an extension of IFS, categorized by a membership function and non-membership function, which are intervals instead of real numbers. Some aggregation operators (Xu, 2007; Xu et al., 2006; Garg, 2016; Garg, 2017; Gou et al., 2017) are the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator have been projected and employed to solve decision-making problems. However, the total membership and non-membership degree may not be limited to 1. For example, if a person gives some priority to an object like 0.7 and a non-priority degree like 0.4 then 0.7+0.4 = 1.1 >1. Therefore, IFS was not eligible to handle this situation. Due to this problem, there was a need for extension and

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hence the concept of Pythagorean fuzzy sets emerged (Yager et al., 2013). PFS tries to solve the problem by giving the place of total condition to the sum of squares because this helps the attainable area to grow (Yager, 2013). It can be seen as $0.7^2+0.4^2 < 1$. Researchers have given many aggregation operators based on PFS (Rahman et al., 2016; Garg, 2017, Lu et al., 2017).

After effective application of PFS, it has been extended to interval-valued PFS (IVPFS) (Zhang, 2016). Numerous weighted average and weighted geometric aggregation operators for the IVPFSs (Rahman et al., 2020) have been established for resolving the decision-making complications. Then (Garg 2019) came out with some new logarithm operational laws (LOL) for the Pythagorean fuzzy sets with real number base λ and operators constructed on these rules. Though, in terms of the information measure concepts, several measures such as novel accuracy function (Garg, 2016), correlation coefficient (Garg, 2016), improved accuracy function (Garg, 2017), and improved score function (Garg, 2017) have been defined under the PFS and IVPFS environment and applied to explain the decision-making problems.

There can be environments where the involvement degree is neither a single value nor an interval, but a set of possible values. Because of this idea, another extension of fuzzy sets came into existence known as hesitant fuzzy sets (HFS) (Torra 2010). In HFS the membership degree can own a set of distinct values lying among [0,1]. To combine various HFSs, specific sorts of hesitant aggregation operators have been evolved by (Xia & Xu, 2011; Zhu et al., 2012) and had been implemented in fixing the decision-making problems. Later on, (Zhu et al., 2012) extended the HFSs to dual HFSs (DHFSs), which contain two sets of characteristic functions whose degrees sum is less than one. (Wang et al., 2014) offered some weighted averaging and geometric aggregation operators under the DHFSs environment.

HFSs and DHFSs have been extensively used by researchers, but they are effective only for those cases in which their corresponding membership degrees sum is less than one. Though, within the actual world, the choice makers may deliver their preferences, in the form of a discrete set, which won't fulfill their conditions. Therefore, these sets are extended to form hesitant Pythagorean fuzzy sets (HPFS), which are a grouping of HFS and PFS. Afterward, many aggregation operators for HPFS have been described. (Lu et al., 2017) established some Hamacher aggregation operators which include the hesitant Pythagorean fuzzy Hamacher weighted average operator, and hesitant Pythagorean fuzzy Hamacher weighted geometric operator. (Khan et al., 2017) projected maximizing deviation technique for HPFNs where facts about attribute weights were inadequate. (Garg, 2018) defined some weighted and geometric aggregation operators for HPFS. Tang et al. (2019) defined some dual hesitant Pythagorean fuzzy Heronian mean aggregation operators such as the dual hesitant Pythagorean fuzzy generalized weighted Heronian mean (DHPFGWHM) operator and the dual hesitant Pythagorean fuzzy generalized geometric weighted Heronian mean (DHPFGGWHM) operator. Yang et al. (2019) developed some hesitant Pythagorean fuzzy interaction weighted Bonferroni mean (HPFIWBM) operator, and the hesitant Pythagorean fuzzy interaction weighted geometric Bonferroni mean (HPFIWGBM) aggregation operator.

Later, (Garg 2020) defined the aggregation operators based on some new kind of processes such as neutrality. Wang et al. (2021) proposed the idea of interactive aggregation operators for Pythagorean fuzzy sets in decision-making problems. (Rahman et al., 2019) developed some generalized Einstein aggregation operators for PFNs. Akram et al. (2020) defined Dombi operators for dealing with the decision-making problems in Pythagorean fuzzy environment. (Sarkar et al. 2021) defined the Archimedean aggregation operators based on Maclaurin symmetric mean for aggregating HPFE in decision-making problems. Senapati et al. (2022) defined Aczel-Alsina aggregation operators for PFS.

Supplier selection has been recognized as a crucial factor for determining a company's reputation and success in extremely competitive markets. Supplier selection requires the evaluation of suppliers based on conflicting attributes and thus can be regarded as an MCDM problem (Vasiljevic et al., 2018). MCDM methods are widely accepted to solve supplier selection problems. Since real-life include ambiguity in decisions, fuzzy MCDM methods (Memari et al., 2019) have attracted researchers for solving supplier selection problem. Luthra et al. (2017) used AHP and VIKOR, Baneian et al. (2018) employed fuzzy grey relational analysis (GRA), Sen et al. (2018) used intuitionistic fuzzy TOPSIS, (Chen et al., 2020) employed DEMATEL and TOPSIS, (Stevic et al., 2020) presented a new methodology MARCOS, Sobhanallahi et al. (2020) considered a fuzzy multi-objective model, Tsai et al. (2021) employed fuzzy DEA, for solving supplier selection problem. Yu et al. (2022) integrates stochastic and fuzzy programming to solve multi-objective optimization models for green supply chain management. Khan et al. (2021) employs HFS, cumulative prospect theory, and VIKOR to find solutions for multi-tier supply chains. The Ordinal Priority approach (OPA) has also been employed by researchers, (Mahmoudi et al., 2021, 2022, 2022) employed Grey OPA, and fuzzy OPA for supplier selection problems. The study incorporates a novel approach in which ambiguity is dealt using sine trigonometric function based operational laws considering the risk preferences of decision makers to efficiently handle the supplier selection problems in HPF environment.

In MCDM problems the weights of attributes play a very important role and during early studies, the easiest way to determine the attribute weights was to take equal weights (Wang et al., 2009). But the final ranking depends on the weights of attributes hence taking equal weights was never an appropriate option (Ginevicius 2011). During further studies, many new weight determination methods were formed which were classified into subjective, objective, and hybrid methods. In subjective methods, weights depend completely on the DM's preferences like the SMART method (Zardari et al., 2015). In objective methods weights depend on the data in the decision matrix like the ENTROPY method, CRITIC method etc. (Wu et al., 2020). Yingming (1997) defined weight determination method based on maximum deviations. Hybrid methods contain a combination of both (Delice et al., 2020; Freeman & Chen, 2015; Du et al., 2020).

The score function helps in ranking the fuzzy sets, to select the better alternative among given by assigning a score value to the alternatives. Various score functions for the HPFS have been defined in the past (Zhu et al., 2012; Khan et al., 2017) but there exist situations where they are unable to rank the alternatives due to similar value of score function or gave an unreasonable ranking of alternatives. Thus, defining new score functions to overcome the shortcomings of previously defined functions has attracted many researchers.

It can be noted that in all the operators defined above operational laws perform a very important role. Various new operational laws can be defined considering different functions. Thus, inspired by the above discussed works, the motive of the study is to define new operational laws based on sine trigonometric function carrying special properties like periodicity, symmetricity, and restricted range, hence compared with previously defined aggregation operators they are more likely to satisfy the decision maker preferences, along with the novel weight determination method and score function to provide a complete algorithm for GDM in HPF environment.

Thus, keeping all the mentioned points in mind, the primary objectives of the study are-

1. To develop new sine trigonometric operational laws for HPFS.
2. To form the aggregation operators based on the defined operational laws.
3. To study the properties of defined operators.
4. To effectively consider the relationship between arguments.
5. To define a z-score measure for determining attribute weights.
6. To define a novel score function to conquer the shortcomings of previously defined score functions in the HPF environment.
7. To develop an effective algorithm for dealing with MAGDM problems using the proposed operators in the HPF environment.
8. To depict the effectiveness of the proposed algorithm using the supplier selection problem.

The rest of the paper is as follows: Section 2 contains a basic definition and properties of hesitant Pythagorean fuzzy sets. In section 3 the multi-attribute group decision-making model is established by defining the operational laws, aggregation operators, weight determination method and score function in the HPF environment. Section 4 applies the proposed approach to supplier selection problem. Section 5 gives the comparative analysis and discussions. Section 6 contains the experimental evaluations. The paper ends with some conclusions in Section 7.

2. Preliminaries

This section consists of some basic definitions related to HPFS on the universal set X.

Definition 2.1. (Garg 2018) A HPFS "D" on X is defined as

$$D = \{< x, h(x), g(x)> \mid x \in X\}$$

where $h(x)$ and $g(x)$ denote the membership and non-membership degrees of the element $x \in X$ to the set D, respectively, following the conditions $0 \leq \gamma, \eta \leq 1$ and $0 \leq (\gamma^+)2, (\eta^+)2 \leq 1$ where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \bigcup_{\gamma \epsilon h(x)} \max\{\gamma\}$ and $\eta^+ \in \bigcup_{\eta \epsilon g(x)} \max\{\eta\}$ for all $x \in X$. Thus, the HPFS can be denoted by the pair d = (h, g).

Definition 2.2. (Garg 2018) The complement of a non-empty HPFE $d = (h; g)$, is defined as-

$$d^c = \begin{cases} \bigcup_{\gamma \in h, \eta \in g} \{\eta, \gamma\} & if g \neq \phi, h \neq \phi \\ \bigcup_{\gamma \in h} \{\sqrt{1-\gamma 2}, \phi\} & if g = \phi, h \neq \phi \\ \bigcup_{\eta \in g} \{\phi, \sqrt{1-\eta 2}\} & if g \neq \phi, h = \phi \end{cases}$$

Definition 2.3. (Garg 2018) For any three HPFEs $d = (h, g)$, $d1 = (h1, g1)$, and $d2 = (h2, g2)$, the basic operations are:

1. $d1 \oplus d2 = \{h1 \oplus g1, h2 \oplus g2\} = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \left\{\sqrt{\gamma_1 2 + \gamma_2 2 - \gamma_1 \gamma_2}, \eta_1 \eta_2\right\}$
2. $d1 \otimes d2 = \{h1 \otimes g1, h2 \otimes g2\} = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \left\{\gamma_1 \gamma_2, \sqrt{\eta_1 2 + \eta_2 2 - \eta_1 \eta_2}\right\}$

3. $\lambda d = \bigcup_{\gamma \in h, \eta \in g} \{\sqrt{1-(1-\gamma^2)^{\lambda}}, \eta^{\lambda}\}$

4. $d^{\lambda} = \bigcup_{\gamma \in h, \eta \in g} \{\gamma^{\lambda}, \sqrt{1-(1-\eta^2)^{\lambda}}\}$

where $\lambda > 0$ is a real number.

Definition 2.4. (Garg 2018) Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any two HPFEs, then the score and accuracy function of d_i ($i = 1, 2$) are given as:

$$S(d_i) = \sum \gamma^2 / *hi \ - \ \sum \eta^2 / *gi \tag{1}$$

$$H(di) = \sum \gamma^2 / *hi \ + \ \sum \eta^2 / *gi \tag{2}$$

where $*hi$ and $*gi$ are the numbers of the elements in hi and gi, respectively.

(1) If $S(d_1) > S(d_2)$, then d_1 is superior to d_2, i.e., $d1 > d2$;
(2) If $S(d_1) = S(d_2)$, then
  (a) If $H(d_1) > H(d_2)$, then d_1 is superior to d_2, i.e., $d1 > d2$.
  (b) If $H(d_1) = H(d_2)$, then d_1 is equivalent to d_2, i.e., $d1 \sim d2$.

Definition 2.5. (Tang et al. 2019) **The Heronian Mean Operator:** Let bi (i = 1, 2, …., n) be a group of non-negative real numbers. Then, the Heronian mean (HM) operator can be defined as:

$$\text{HM}(b_1, b_2 \dots b_n) = \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} (b_i b_j)^{\frac{1}{2}} \tag{3}$$

Definition 2.6. (Tang et al., 2019) Assume that b_i (i = 1, 2, ……, n) are a group of nonnegative real numbers and m, n > 0. Then, the GHM operator can be defined as:

$$\text{GHM}(b_1, b_2 \dots b_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} {b_i}^{m} \, {b_j}^{n}\right)^{\frac{1}{(m+n)}} \tag{4}$$

When m = n = 1/2, the GHM operator will reduce to the Heronian mean (HM) operator. Thus, the HM operator is a special case of the GHM operator.

3. A MAGDM model based on trigonometric aggregation operators

This section, introduces a MAGDM model based on new Trigonometric aggregation operators for the HPFNs. The flow chart of the model is shown in Fig. 1 and all the abbreviations used in the paper are listed in the form of table in appendix A.

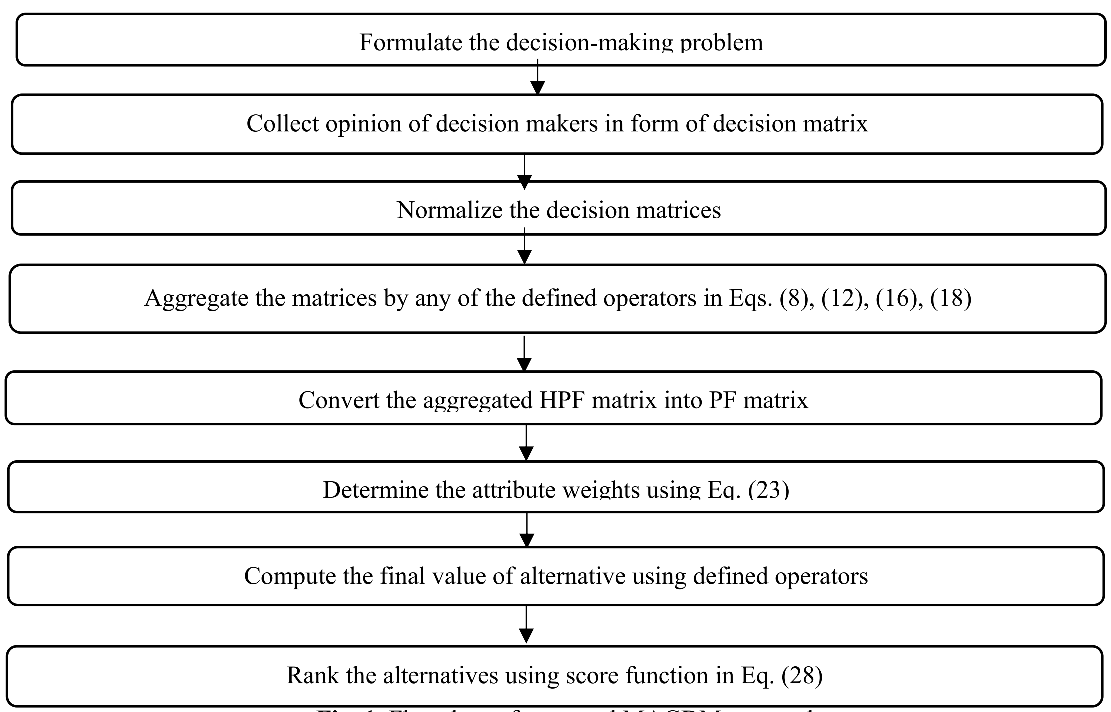

Fig. 1. Flowchart of proposed MAGDM approach

3.1. Trigonometric Operational Laws (TOLs)

Definition 3.1.1. Let X be a non-empty set and $D = \{< x, h(x), g(x)> \mid x \in X\}$ be a HPFS, we define TOL of HPFS as-

$$\sin_{\lambda} D = \{x, \sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}}, \sqrt{\sin\lambda\eta} \mid x \in X\} \tag{5}$$

where, $0 \le \lambda \le 90$

it is clearly seen that $\sin_{\lambda} D$ is also a HPFS.

Proof: we assume $\sin_{\lambda} D = (h_D, g_D)$, so we need to show that $0 \le \gamma_D, \eta_D \le 1$ and $0 \le (\gamma_D{}^{+})^2, (\eta_D{}^{+})^2 \le 1$

since D is a HPFS we have $0 \le \gamma, \eta \le 1$ and $0 \le (\gamma^{+})^2, (\eta^{+})^2 \le 1$

now since $0 \le \gamma \le 1 \Rightarrow 0 \le \gamma^2 \le 1 \Rightarrow 0 \le \sqrt{(1-\gamma^2)} \le 1 \Rightarrow 0 \le \lambda\sqrt{(1-\gamma^2)} \le 90$

so, $0 \le \sin\lambda\sqrt{(1-\gamma^2)} \le 90 \Rightarrow 0 \le \sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}} \le 1 \Rightarrow 0 \le \gamma_D \le 1$

in a similar way, since $0 \le \eta \le 1 \Rightarrow 0 \le \lambda\eta \le 90$

so, $0 \le \sin\lambda\eta \le 1 \Rightarrow 0 \le \sqrt{\sin\lambda\eta} \le 1 \Rightarrow 0 \le \eta_D \le 1$

finally, $0 \le (\gamma_D{}^{+})^2, (\eta_D{}^{+})^2 = 1 - \sin\lambda\sqrt{(1-\gamma_D{}^{+2})} + \sin\lambda\,\eta_D{}^{+} \le 1 - \sin\lambda\,\eta_D{}^{+} + \sin\lambda\,\eta_D{}^{+} = 1$

hence, $\sin_{\lambda} D$ is also a HPFS.

Definition 3.1.2. Let d = (h, g) be a HPFN then

$$\sin_{\lambda} D = \bigcup_{\gamma\in h,\eta\in g} \left\{\sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}}, \sqrt{\sin\lambda\eta}\right\} \qquad 0 \le \lambda \le 90 \tag{6}$$

$\sin_{\lambda} D$ is a trigonometric operator and the value of $\sin_{\lambda} D$ is called as trigonometric HPFN (THPFN).

Now we will discuss some basic properties of THPFN based on TOL.

Theorem 3.1.1. Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any two HPFEs, then

- **a)** $\sin_{\lambda} d_1 \oplus \sin_{\lambda} d_2 = \sin_{\lambda} d_2 \oplus \sin_{\lambda} d_1$
- **b)** $\sin_{\lambda} d_1 \otimes \sin_{\lambda} d_2 = \sin_{\lambda} d_2 \otimes \sin_{\lambda} d_1$
- **c)** $(\sin_{\lambda} d_1 \oplus \sin_{\lambda} d_2) \oplus \sin_{\lambda} d_3 = \sin_{\lambda} d_1 \oplus (\sin_{\lambda} d_2 \oplus \sin_{\lambda} d_3)$
- **d)** $(\sin_{\lambda} d_1 \otimes \sin_{\lambda} d_2) \otimes \sin_{\lambda} d_3 = \sin_{\lambda} d_1 \otimes (\sin_{\lambda} d_2 \otimes \sin_{\lambda} d_3)$

Proof: Straight forward from definition 3.1.2

Theorem 3.1.2. Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any two HPFEs, and k, k_1, $k_2 > 0$ be three real numbers then

- **a)** $k(\sin_{\lambda} d_1 \oplus \sin_{\lambda} d_2) = k \sin_{\lambda} d_1 \oplus k \sin_{\lambda} d_2$
- **b)** $(\sin_{\lambda} d_1 \otimes \sin_{\lambda} d_2)^{k} = \sin_{\lambda} d_1{}^{k} \otimes \sin_{\lambda} d_2{}^{k}$
- **c)** $k_1 \sin_{\lambda} d_1 \oplus k_2 \sin_{\lambda} d_1 = (k_1 + k_2) \sin_{\lambda} d_1$
- **d)** $(\sin_{\lambda} d_1)^{k_1} \otimes (\sin_{\lambda} d_1)^{k_2} = (\sin_{\lambda} d_1)^{k_1+k_2}$
- **e)** $((\sin_{\lambda} d_1)^{k_1})^{k_2} = (\sin_{\lambda} d_1)^{k_1 k_2}$

Theorem 3.1.3. Let $d = (h, g)$ be a HPFN and $0 \le \lambda_1 \le \lambda_2 \le 1$, then $\sin_{\lambda_1} d \ge \sin_{\lambda_2} d$

Theorem 3.1.4. Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any two HPFEs, and $0 \le \lambda \le 90$, such that $\gamma_1 \in h_1 \le \gamma_2 \in h_2$, $\eta_1 \in g_1 \ge \eta_2 \in g_2$ i.e., $d_1 \le d_2$ then $\sin_{\lambda} d_1 \le \sin_{\lambda} d_2$

Proof: proof of theorem 3.1.2, 3.1.3 and 3.1.4 is specified in appendix B.

3.2. Aggregation operators

Based on TOL of HPFNs we define following weighted and geometric aggregation operators.

Definition 3.2.1. Let $d_i = (h_i, g_i)$ (i=1,2,3…n) be the collection of HPFNs and $0 \le \lambda_i \le 90$ then THPFWA: $\Omega^n \to \Omega$ is defined as-

$$\text{THPFWA}(d_1, d_2 \ldots\ldots d_n) = \omega_1 \sin_{\lambda_1} d1 \oplus \omega_2 \sin_{\lambda_2} d2 \ldots\ldots\oplus \omega_n \sin_{\lambda_n} dn \tag{7}$$

where $\omega = (\omega_1, \omega_2 \cdots \omega_n)^T$ be weight vector with $\omega_i \ge 0$ and $\sum_{i=1}^{n} \omega_i = 1$

Theorem 3.2.1. For the collection of HPFNs $d_i = (h_i, g_i)$ (i=1,2,3…n) the aggregated value by using definition 3.2.1 is still a HPFN and is given by-

THPFWA ($d_1, d_2 \ldots\ldots d_n$) =

$$\bigcup_{\gamma_1\epsilon h_1,\gamma_2\epsilon h_2\ldots\ldots\ \gamma_n\epsilon h_n, n_1\epsilon g_1, n_2\epsilon g_2\ldots\ldots n_n\epsilon g_n} \left\{\left\{\sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda_i\sqrt{1-\gamma_i^2}\right]^{\omega_i}}\right\},\left\{\prod_{i=1}^{n}\sqrt{(\sin\lambda_i\eta_i)}^{\omega_i}\right\}\right\} \tag{8}$$

Proof: It can be proved by mathematical induction on n. Since for each i $d_i = (h_i, g_i)$ is a HPFE which implies that $0 \le \gamma, \eta \le 1$ and $0 \le (\gamma^+)^2, (\eta^+)^2 \le 1$ where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \bigcup_{\gamma\epsilon h(x)} \max\{\gamma\}$ and $\eta^+ = \bigcup_{\eta\epsilon g(x)} \max\{\eta\}$.

Then the following steps of mathematical induction have been followed.

Step.1. When n=2, $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ thus by operations of HPFS

$$\omega_1 \sin\lambda_1\, d_1 = \bigcup_{\gamma_1\epsilon h_1,\eta_1\epsilon g_1}\left\{\sqrt{(\sin\lambda_1\sqrt{(1-\gamma_1^2)}\,)^{\omega_1}}, \sqrt{\sin\lambda_1\eta_1{}^{\omega_1}}\right\}$$

$$\omega_2 \sin\lambda_2\, d_2 = \bigcup_{\gamma_1\epsilon h_1,\eta_1\epsilon g_1}\left\{\sqrt{(\sin\lambda_2\sqrt{(1-\gamma_2^2)}\,)^{\omega_2}}, \sqrt{\sin\lambda_2\eta_2{}^{\omega_2}}\right\}$$

THPFWA (d_1, d_2) = $\omega_1\ \sin\lambda_1\ d1 \oplus \omega_2\ \sin\lambda_2\ d2$

$$= \bigcup_{\gamma_1\epsilon h_1,\gamma_2\epsilon h_2,n_1\epsilon g_1,n_2\epsilon g_2}\left\{\left\{\sqrt{1-\left[\sin\lambda_1\sqrt{1-\gamma_1^2}\right]^{\omega_1}+1-\left[\sin\lambda_2\sqrt{1-\gamma_2^2}\right]^{\omega_2}-\left(1-\left[\sin\lambda_1\sqrt{1-\gamma_1^2}\right]^{\omega_1}\right).(1-\left[\sin\lambda_2\sqrt{1-\gamma_2^2}\right]^{\omega_2})}\right\},\left\{\sqrt{(\sin\lambda_1\eta_1)}^{\omega_1}.\sqrt{(\sin\lambda_2\eta_2)}^{\omega_2}\right\}\right\}$$

$$= \bigcup_{\gamma_1\epsilon h_1,\gamma_2\epsilon h_2,n_1\epsilon g_1,n_2\epsilon g_2.}\left\{\left\{\sqrt{1-\prod_{i=1}^{2}\left[\sin\lambda_i\sqrt{1-\gamma_i^2}\right]^{\omega_i}}\right\},\left\{\prod_{i=1}^{2}\sqrt{(\sin\lambda_i\eta_i)}^{\omega_i}\right\}\right\}$$

Hence the result is true for n=2

Step.2. Assume the result is true for n=k

THPFWA ($d_1, d_2 \ldots\ldots d_k$) =

$$\bigcup_{\gamma_1\epsilon h_1,\gamma_2\epsilon h_2\ldots\ldots\ \gamma_k\epsilon h_k, n_1\epsilon g_1, n_2\epsilon g_2\ldots\ldots n_k\epsilon g_k} \left\{\left\{\sqrt{1-\prod_{i=1}^{k}\left[\sin\lambda_i\sqrt{1-\gamma_i^2}\right]^{\omega_i}}\right\},\left\{\prod_{i=1}^{k}\sqrt{(\sin\lambda_i\eta_i)}^{\omega_i}\right\}\right\}$$

Step.3. When n=k+1 we have

THPFWA ($d_1, d_2 \ldots\ldots d_{k+1}$) = THPFWA ($d_1, d_2 \ldots\ldots d_k$) $\oplus\ \omega_{k+1} \sin_\lambda\ d_{k+1}$

$$= \bigcup_{\gamma_1\epsilon h_1,\gamma_2\epsilon h_2\ldots\ldots\ \gamma_k\epsilon h_k, n_1\epsilon g_1, n_2\epsilon g_2\ldots\ldots n_k\epsilon g_k} \left\{\left\{\sqrt{1-\prod_{i=1}^{k}\left[\sin\lambda_i\sqrt{1-\gamma_i^2}\right]^{\omega_i}}\right\},\left\{\prod_{i=1}^{k}\sqrt{(\sin\lambda_i\eta_i)}^{\omega_i}\right\}\right\} \oplus$$

$$\bigcup_{\gamma_{k+1}\epsilon h_{k+1},\eta_{k+1}\epsilon g_{k+1}}\left\{\sqrt{(\sin\lambda\sqrt{(1-\gamma_{k+1}{}^2)}\,)^{\omega_{k+1}}}, \sqrt{\sin\lambda\eta_{k+1}{}^{\omega_{k+1}}}\right\}$$

$$= \bigcup_{\gamma_1\epsilon h_1,\gamma_2\epsilon h_2\ldots\ldots\ \gamma_{k+1}\epsilon h_{k+1}, n_1\epsilon g_1, n_2\epsilon g_2\ldots\ldots n_{k+1}\epsilon g_{k+1}} \left\{\left\{\sqrt{1-\prod_{i=1}^{k+1}\left[\sin\lambda_i\sqrt{1-\gamma_i^2}\right]^{\omega_i}}\right\},\left\{\prod_{i=1}^{k+1}\sqrt{(\sin\lambda_i\eta_i)}^{\omega_i}\right\}\right\}$$

Thus, the result holds for n=k+1 also, hence by mathematical induction theorem 3.2.1 holds for every n.

Next to show that the aggregated value by THPFWA is again a HPFE we assume that

THPFWA ($d_1, d_2 \ldots\ldots d_n$) = $\cup\{\gamma_k, \eta_k\}$ where $\gamma_k = \left\{\sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda_i\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}}\right\}$ and

$\eta_k = \left\{\prod_{i=1}^{n}\sqrt{(\sin\lambda_i\eta_i)}^{\omega_i}\right\}$ then we need to show that $0\le \gamma_k \le 1$, $0\le \eta_k \le 1$ and $\gamma_k 2 + \eta_k 2 \le 1$

Since each $d_i = (h_i, g_i)$ is a HPFE which implies that $0 \le \gamma_i , \eta_i \le 1$ and $0 \le (\gamma_i)2 + (\eta_i)2 \le 1$ where $\gamma_i \in h_i$, $\eta_i \in g_i$

Therefore $0 \le 1-(\gamma_i)2 \le 1 \Rightarrow 0 \le \sqrt{1-\gamma_i{}^2} \le 1 \Rightarrow 0 \le \lambda_i\sqrt{1-\gamma_i{}^2} \le 90$

thus, $0 \le \sin\lambda_i\sqrt{1-\gamma_i{}^2} \le 1$ and for $\omega_i \ge 0$ we have $0 \le (\sin\lambda_i\sqrt{1-\gamma_i{}^2})^{\omega_i} \le 1$

$\Rightarrow 0 \le \sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda_i\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}} \le 1 \Rightarrow 0\le \gamma_k \le 1$

Similarly, $0 \le \prod_{i=1}^{n}\sqrt{(\sin\lambda_i\eta_i)}^{\omega_i} \le 1 \Rightarrow 0\le \eta_k \le 1$

Finally, $\gamma_k 2+\eta_k 2 = 1-\prod_{i=1}^{n}\left[\sin\lambda_i\sqrt{1-\gamma_i{}^2}\right]^{\omega_i} + \prod_{i=1}^{n}[\sin\lambda_i\,\eta_i]^{\omega_i} \le 1-\prod_{i=1}^{n}[\sin\lambda_i\,\eta_i]^{\omega_i} + \prod_{i=1}^{n}[\sin\lambda_i\,\eta_i]^{\omega_i} = 1$

From $0 \le (\gamma_i)2 + (\eta_i)2 \le 1$ we have $(\eta_i)2 \le 1 - (\gamma_i)2 \Rightarrow \eta_i \le \sqrt{1-(\gamma_i)2}$

$\Rightarrow \sin\eta_i \le \sin\sqrt{1-(\gamma_i)2} \Rightarrow \prod_{i=1}^{n}[\sin\lambda_i\,\eta_i]^{\omega_i} \le \prod_{i=1}^{n}\left[\sin\lambda_i\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}$

Hence, the aggregated value by THPFWA is again a HPFE.

Example 3.2.1. Consider two HPFEs $d_1 = \{\{0.5\}, \{0.6\}\}$ and $d_1 = \{\{0.3\}, \{0.6, 0.7\}\}$. Consider the importance of these elements as $\omega = (0.6, 0.4)$ and $\lambda_1 = \lambda_2 = 30$. Then by using eq. (7) we have:

$$\text{THPFWA}\,(d1, d2) = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,n_1\in g_1,n_2\in g_2}\left\{\left\{\sqrt{1-\prod_{i=1}^{2}\left[\sin\lambda_i\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}}\right\},\left\{\prod_{i=1}^{2}\sqrt{(\sin\lambda_i\eta_i)}^{\omega_i}\right\}\right\}$$

$$=\left\{\sqrt{1-\left(\sin\lambda_1\sqrt{1-0.5^2}\right)^{0.6} * \left(\sin\lambda_2\sqrt{1-0.3^2}\right)^{0.4}}\right\},\left\{\sqrt{\sin 0.6\lambda_1}^{0\cdot 6} * \sqrt{\sin 0.6\lambda_2}^{0\cdot 4}, \sqrt{\sin 0.6\lambda_1}^{0\cdot 6} * \sqrt{\sin 0.7\lambda_2}^{0\cdot 4}\right\}$$

$$= \left\{\sqrt{1-0\cdot 60926\times 0\cdot 74481}\right\},\{0\cdot 70296\times 0\cdot 79059,\ 0.70296\times 0.81437\}$$

$= \{\{0.73906\}, \{0.55575, 0.57247\}\}$

The THPFWA operator has the following properties.

Property 3.2.1. If all HPFNs are equal i.e., if $d_i = (h_i, g_i) = d$ then THPFWA ($d_1, d_2 \ldots\ldots d_n$) = $\sin_\lambda d$

Property 3.2.2. If $d_i = (h_i, g_i)$ (i=1, 2, …. n), $d^+ = \langle \max_j h_j, \min_j g_j\rangle$ and $d^- = \langle \min_j h_j, \max_j g_j\rangle$ be HPFNs then

$$\sin_\lambda d^- \le \text{THPFWA}\,(d_1, d_2 \ldots\ldots d_n) \le \sin_\lambda d^+$$

Property 3.2.3. Let $d_i = (h_i, g_i)$ and $d_i^* = (h_i^*, g_i^*)$ (i=1, 2, …. n) be two collections of HPFNs. If $\gamma_i \le \gamma_i^*$ and $\eta_i \ge \eta_i^*$, then THPFWA ($d_1, d_2 \ldots\ldots d_n$) $\le$ THPFWA ($d_1^*, d_2^* \ldots\ldots d_n^*$)

Proof: proof of property 3.2.1, 3.2.2 and 3.2.3 is specified in appendix C.

Definition 3.2.2. Let $d_i = (h_i, g_i)$ (i=1,2,3…n) be the collection of HPFNs and $0 \le \lambda \le 90$ then Trigonometric HPFN ordered weighted averaging operator THPFOWA: $\Omega^n \to \Omega$ is defined as:

$$\text{THPFOWA}\,(d_1, d_2 \ldots\ldots d_n) = \omega_1 \sin_{\lambda_{\sigma(1)}} d_{\sigma(1)} \oplus \omega_2 \sin_{\lambda_{\sigma(2)}} d_{\sigma(2)} \ldots\ldots\oplus \omega_n \sin_{\lambda_{\sigma(n)}} d_{\sigma(n)} \quad (9)$$

where $\omega = (\omega_1, \omega_2 \cdots \omega_n)^T$ be weight vector with $\omega_i \ge 0$ and $\sum_{i=1}^{n}\omega_i = 1$, and σ is the permutation of (1, 2……n) such that $d_{\sigma(i-1)} \ge d_{\sigma(i)}$ for i = 2, 3…, n.

Theorem 3.2.2. For the collection of HPFNs $d_i = (h_i, g_i)$ (i=1,2,3…n) the aggregated value by using definition 3.2.2 is still a HPFN and is given by-

THPFOWA ($d_1, d_2 \ldots\ldots d_n$) =

$$\bigcup_{\gamma_{\sigma(i)}\epsilon h_{\sigma(i)}, n_{\sigma(i)}\epsilon g_{\sigma(i)}} \left\{ \left\{ \sqrt{1-\prod_{i=1}^{n}\left[\sin \lambda_{\sigma(i)} \sqrt{1-\gamma_{\sigma(i)}{}^{2}}\right]^{\omega_i}} \right\}, \left\{ \prod_{i=1}^{n} \sqrt{(\sin \lambda_{\sigma(i)} \eta_{\sigma(i)})}^{\omega_i} \right\} \right\} \tag{10}$$

Proof: Proof of this theorem is similar to theorem 3.2.1.

Definition 3.2.3. Let $d_i = (h_i, g_i)$ (i=1,2,3…n) be the collection of HPFNs and $0 \le \lambda_i \le 90$ then Trigonometric HPFN weighted geometric operator THPFWG: $\Omega^n \to \Omega$ is defined as:

$$\text{THPFWG}(d_1, d_2 \ldots\ldots d_n) = (\sin\lambda_1\ d1)^{\omega_1} \otimes (\sin\lambda_2\ d2)^{\omega_2} \ldots\ldots \otimes (\sin\lambda_n\ dn)^{\omega_n} \tag{11}$$

where $\omega = (\omega_1, \omega_2 \cdots \omega_n)^T$ be weight vector with $\omega_i \ge 0$ and $\sum_{i=1}^{n} \omega_i = 1$

Theorem 3.2.3. For the collection of HPFNs $d_i = (h_i, g_i)$ (i=1,2,3…n) the aggregated value by using definition 3.2.3 is still a HPFN and is given by-

THPFWG ($d_1, d_2 \ldots\ldots d_n$) =

$$\bigcup_{\gamma_1\epsilon h_1, \gamma_2\epsilon h_2 \ldots\ldots \gamma_n\epsilon h_n, n_1\epsilon g_1, n_2\epsilon g_2 \ldots\ldots n_n\epsilon g_n} \left\{ \left\{ \prod_{i=1}^{n} \left\{ \sqrt{1-\left[\sin\lambda_i \sqrt{1-\gamma_i{}^2}\right]} \right\}^{\omega_i} \right\}, \left\{ \sqrt{1-\prod_{i=1}^{n}(1-\sin\lambda_i \eta_i)^{\omega_i}} \right\} \right\} \tag{12}$$

Proof: Proof of this theorem is similar to theorem 3.2.1.

Definition 3.2.4. Let $d_i = (h_i, g_i)$ (i=1,2,3…n) be the collection of HPFNs and $0 \le \lambda \le 90$ then Trigonometric

HPFN ordered weighted geometric operator THPFOWG: $\Omega^n \to \Omega$ is defined as:

$$\text{THPFOWG}(d_1, d_2 \ldots\ldots d_n) = \sin_{\lambda_{\sigma(1)}} d_{\sigma(1)}{}^{\omega_1} \oplus \sin_{\lambda_{\sigma(2)}} d_{\sigma(2)}{}^{\omega_2} \ldots\ldots\ldots \oplus \sin_{\lambda_{\sigma(n)}} d_{\sigma(n)}{}^{\omega_n} \tag{13}$$

where $\omega = (\omega_1, \omega_2 \cdots \omega_n)^T$ be weight vector with $\omega_i \ge 0$ and $\sum_{i=1}^{n} \omega_i = 1$, and σ is the permutation of (1, 2……n) such that $d_{\sigma(i-1)} \ge d_{\sigma(i)}$ for i = 2, 3…, n.

Theorem 3.2.4. For the collection of HPFNs $d_i = (h_i, g_i)$ (i=1,2,3…n) the aggregated value by using definition 3.2.4 is still a HPFN and is given by-

THPFOWG ($d_1, d_2 \ldots\ldots d_n$) =

$$\bigcup_{\gamma_{\sigma(i)}\epsilon h_{\sigma(i)}, n_{\sigma(i)}\epsilon g_{\sigma(i)}} \left\{ \left\{ \prod_{i=1}^{n} \left\{ \sqrt{1-\left[\sin_{\lambda_{\sigma(i)}} \sqrt{1-\gamma_{\sigma(i)}{}^2}\right]} \right\}^{\omega_i} \right\}, \left\{ \sqrt{1-\prod_{i=1}^{n}(1-\sin_{\lambda_{\sigma(i)}} \eta_{\sigma(i)})^{\omega_i}} \right\} \right\} \tag{14}$$

Proof: Proof of this theorem is similar to theorem 3.2.1.

As similar to THPFWA operator, the THPFOWA, THPFOWG, and THPFWG operators also holds the same properties.

3.3. Trigonometric Hesitant Pythagorean Fuzzy Heronian Mean Operators

This section combines the above defined operators with the GHM operator to measure the relationship between arguments.

Definition 3.3.1. Let $d_i = (h_i, g_i)$ (i=1,2,3…n) be the collection of HPFNs and $0 \le \lambda \le 90$ then Trigonometric HPFN Generalized heronian weighted averaging operator THPFGHWA: $\Omega^n \to \Omega$ is defined as:

$$\text{THPFGHWA}(d_1, d_2 \ldots\ldots d_n) = \frac{2}{n(n+1)} \left[\oplus_{i=1}^{n} \oplus_{j=1}^{n} (\omega_i \sin\lambda_i\ d_i)^m \otimes (\omega_j \sin\lambda_i\ d_j)^n\right]^{\frac{1}{(m+n)}} \tag{15}$$

where $\omega = (\omega_1, \omega_2 \cdots \omega_n)^T$ be weight vector with $\omega_i \ge 0$ and $\sum_{i=1}^{n} \omega_i = 1$

Theorem 3.3.1. For the collection of HPFNs $d_i = (h_i, g_i)$ (i=1,2,3…n) the aggregated value by using definition 3.3.1 is still a HPFN and is given by-

THPFGHWA ($d_1, d_2 \ldots\ldots d_n$) =

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\left[\sqrt{1-\prod_{i=1,j=i}^{n}\left[1-\left\{1-\left(\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\omega_i}\right\}^m.\left\{1-\left(\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\omega_j}\right\}^n\right]^{\frac{2}{n(n+1)}}}\right]^{\frac{1}{(m+n)}}\right\},\\ \left\{\sqrt{1-\left\{1-\prod_{i=1,j=i}^{n}\left[1-\left(1-\left(\sin\lambda_i\eta_i\right)^{\omega_i}\right)^m.\left(1-\left(\sin\lambda_j\eta_j\right)^{\omega_j}\right)^n\right]^{\frac{2}{n(n+1)}}\right\}^{\frac{1}{(m+n)}}}\right\}\end{array}\right\} \quad (16)$$

Proof: $\omega_i \sin\lambda_i\, d_i = \bigcup_{\gamma_i\in h_i,\eta_i\in g_i}\left\{\sqrt{(\sin\lambda\sqrt{(1-\gamma_i^2)}\,)^{\omega_i}}, \sqrt{\sin\lambda\eta_i^{\omega_i}}\right\}$

$\omega_j \sin\lambda_j\, d_j = \bigcup_{\gamma_j\in h_j,\eta_j\in g_j}\left\{\sqrt{(\sin\lambda\sqrt{(1-\gamma_j^2)}\,)^{\omega_j}}, \sqrt{\sin\lambda\eta_j^{\omega_j}}\right\}$

$(\omega_i\ \sin\lambda_i\ di)^m = \bigcup_{\gamma_i\in h_i,\eta_i\in g_i}\left\{\left[\sqrt{(\sin\lambda\sqrt{(1-\gamma_i^2)}\,)^{\omega_i}}\right]^m, \left[\sqrt{1-(1-\sin\lambda\eta_i^{\omega_i})^m}\right]\right\}$

$(\omega_j\ \sin\lambda_j\ dj)^n = \bigcup_{\gamma_j\in h_j,\eta_j\in g_j}\left\{\left[\sqrt{(\sin\lambda\sqrt{(1-\gamma_j^2)}\,)^{\omega_j}}\right]^n, \left[\sqrt{1-(1-\sin\lambda\eta_j^{\omega_j})^n}\right]\right\}$

$(\omega_i \sin\lambda_i\ d_i)^m \otimes (\omega_j \sin\lambda_j\ d_j)^n =$

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\left(\sqrt{1-((\sin\lambda_i\sqrt{(1-\gamma_i^2)}\,)^{\omega_i}}\right)^m.\left(\sqrt{1-((\sin\lambda_j\sqrt{(1-\gamma_j^2)}\,)^{\omega_j}}\right)^n\right\},\\ \left\{\sqrt{1-(1-\sin\lambda_i\eta_i^{\omega_i})^m.(1-\sin\lambda_j\eta_j^{\omega_j})^n}\right\}\end{array}\right\}$$

$\left[\oplus_{i=1}^{n}\oplus_{j=1}^{n}(\omega_i \sin\lambda_i\ d_i)^m \otimes (\omega_j \sin\lambda_j\ d_j)^n\right] =$

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\left[\sqrt{1-\prod_{i=1,j=i}^{n}\left[1-\left\{1-\left(\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\omega_i}\right\}^m.\left\{1-\left(\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\omega_j}\right\}^n\right]}\right]\right\},\\ \left\{\sqrt{\prod_{i=1,j=i}^{n}\left[1-\left(1-\left(\sin\lambda_i\eta_i\right)^{\omega_i}\right)^m.\left(1-\left(\sin\lambda_j\eta_j\right)^{\omega_j}\right)^n\right]}\right\}\end{array}\right\}$$

$\frac{2}{n(n+1)}\left[\oplus_{i=1}^{n}\oplus_{j=1}^{n}(\omega_i \sin\lambda_i\ d_i)^m \otimes (\omega_j \sin\lambda_j\ d_j)^n\right] =$

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\sqrt{1-\prod_{i=1,j=i}^{n}\left[1-\left\{1-\left(\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\omega_i}\right\}^m.\left\{1-\left(\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\omega_j}\right\}^n\right]^{\frac{2}{n(n+1)}}}\right\},\\ \left\{\sqrt{\prod_{i=1,j=i}^{n}\left[1-\left(1-\left(\sin\lambda_i\eta_i\right)^{\omega_i}\right)^m.\left(1-\left(\sin\lambda_j\eta_j\right)^{\omega_j}\right)^n\right]^{\frac{2}{n(n+1)}}}\right\}\end{array}\right\}$$

THPFGHWA ($d_1, d_2 \ldots\ldots d_n$) $= \frac{2}{n(n+1)}\left[\oplus_{i=1}^{n}\oplus_{j=1}^{n}(\omega_i \sin\lambda_i\ d_i)^m \otimes (\omega_j \sin\lambda_j\ d_j)^n\right]^{\frac{1}{(m+n)}} =$

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\left[\sqrt{1-\prod_{i=1,j=i}^{n}\left[1-\left\{1-\left(\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\omega_i}\right\}^m.\left\{1-\left(\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\omega_j}\right\}^n\right]^{\frac{2}{n(n+1)}}}\right]^{\frac{1}{(m+n)}}\right\},\\ \left\{\sqrt{1-\left\{1-\prod_{i=1,j=i}^{n}\left[1-\left(1-\left(\sin\lambda_i\eta_i\right)^{\omega_i}\right)^m.\left(1-\left(\sin\lambda_j\eta_j\right)^{\omega_j}\right)^n\right]^{\frac{2}{n(n+1)}}\right\}^{\frac{1}{(m+n)}}}\right\}\end{array}\right\}$$

Definition 3.3.2. Let $d_i = (h_i, g_i)$ (i=1,2,3…n) be the collection of HPFNs and $0 \le \lambda_i \le 90$ then Trigonometric HPFN Generalized heronian weighted geometric operator THPFGHWG: $\Omega^n \to \Omega$ is defined as

$$\text{THPFGHWG}\ (d_1, d_2 \ldots\ldots d_n) = \frac{1}{(m+n)}\left[\otimes_{i=1}^{n}\otimes_{j=1}^{n} m(\sin\lambda_i\ d_i)^{\omega_i} \oplus n(\sin\lambda_j\ d_j)^{\omega_j}\right]^{\frac{2}{n(n+1)}} \quad (17)$$

where $\omega = (\omega_1, \omega_2 \cdots \omega_n)^T$ be weight vector with $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$

Theorem 3.3.2. For the collection of HPFNs $d_i = (h_i, g_i)$ (i=1,2,3…n) the aggregated value by using definition 3.3.2 is still a HPFN and is given by-

THPFGHWG ($d_1, d_2 \ldots\ldots d_n$) =

$$\bigcup_{\gamma_i \epsilon h_i, \gamma_j \epsilon h_j, n_i \epsilon g_i, n_j \epsilon g_j} \left\{ \begin{array}{c} \left\{ \left(\sqrt{1 - \left[1 - \prod_{i=1, j=i}^{n} \{1 - [1 - (1 - \sin \lambda_i \sqrt{1-\gamma_i^2})^{\omega_i}]^m . [1 - (1 - \sin \lambda_j \sqrt{1-\gamma_j^2})^{\omega_j}]^n\}^{\frac{2}{n(n+1)}}\right]^{\frac{1}{(m+n)}}} \right) \right\}, \\ \left\{ \left[\sqrt{1 - \prod_{i=1, j=i}^{n} [1 - \{1 - (1 - \sin \lambda_i \eta_i)^{\omega_i}\}^m . \{1 - (1 - \sin \lambda_j \eta_j)^{\omega_j}\}^n]^{\frac{2}{n(n+1)}}} \right]^{\frac{1}{(m+n)}} \right\} \end{array} \right\} \quad (18)$$

Proof: $(\sin \lambda_i \ di)^{\omega_i} = \bigcup_{\gamma_i \epsilon h_i, \eta_i \epsilon g_i} \left\{ \left(\sqrt{1 - (\sin \lambda_i \sqrt{(1-\gamma_i^2)}} \right)^{\omega_i}, \sqrt{1 - (1 - \sin \lambda_j \eta_i)^{\omega_i}} \right\}$

$$(\sin \lambda_j \ dj)^{\omega_j} = \bigcup_{\gamma_j \epsilon h_j, \eta_j \epsilon g_j} \left\{ \left(\sqrt{1 - (\sin \lambda_i \sqrt{(1-\gamma_j^2)}} \right)^{\omega_j}, \sqrt{1 - (1 - \sin \lambda_j \eta_j)^{\omega_j}} \right\}$$

$$m.(\sin \lambda_i \ di)^{\omega_i} = \bigcup_{\gamma_i \epsilon h_i, \eta_i \epsilon g_i} \left\{ \left[\sqrt{1 - [1 - (1 - \sin \lambda_i \sqrt{1-\gamma_i^2})^{\omega_i}]^m} \right], \left[\sqrt{1 - (1 - \sin \lambda_j \eta_i)^{\omega_i}} \right]^m \right\}$$

$$n.(\sin \lambda_j \ dj)^{\omega_j} = \bigcup_{\gamma_j \epsilon h_j, \eta_j \epsilon g_j} \left\{ \left[\sqrt{1 - [1 - (1 - \sin \lambda_i \sqrt{1-\gamma_j^2})^{\omega_j}]^n} \right], \left[\sqrt{1 - (1 - \sin \lambda_j \eta_j)^{\omega_j}} \right]^n \right\}$$

$m(\sin \lambda_i \ d_i)^{\omega_i} \oplus n(\sin \lambda_j \ d_j)^{\omega_j} =$

$$\bigcup_{\gamma_i \epsilon h_i, \gamma_j \epsilon h_j, n_i \epsilon g_i, n_j \epsilon g_j} \left\{ \begin{array}{c} \left\{ \sqrt{1 - [1 - (1 - \sin \lambda_i \sqrt{1-\gamma_i^2})^{\omega_i}]^m . [1 - (1 - \sin \lambda_j \sqrt{1-\gamma_j^2})^{\omega_j}]^n} \right\}, \\ \left\{ \sqrt{\{1 - (1 - \sin \lambda_i \eta_i)^{\omega_i}\}^m . \{1 - (1 - \sin \lambda_j \eta_j)^{\omega_j}\}^n} \right\} \end{array} \right\}$$

$\otimes_{i=1}^{n} \otimes_{j=1}^{n} m(\sin \lambda_i \ d_i)^{\omega_i} \oplus n(\sin \lambda_j \ d_j)^{\omega_j} =$

$$\bigcup_{\gamma_i \epsilon h_i, \gamma_j \epsilon h_j, n_i \epsilon g_i, n_j \epsilon g_j} \left\{ \begin{array}{c} \left\{ \left(\sqrt{\prod_{i=1, j=i}^{n} [1 - [1 - (1 - \sin \lambda_i \sqrt{1-\gamma_i^2})^{\omega_i}]^m . [1 - (1 - \sin \lambda_j \sqrt{1-\gamma_j^2})^{\omega_j}]^n]} \right) \right\}, \\ \left\{ \sqrt{1 - \prod_{i=1, j=i}^{n} [1 - \{1 - (1 - \sin \lambda_i \eta_i)^{\omega_i}\}^m . \{1 - (1 - \sin \lambda_j \eta_j)^{\omega_j}\}^n]} \right\} \end{array} \right\}$$

$\left[\otimes_{i=1}^{n} \otimes_{j=1}^{n} m(\sin \lambda_i \ d_i)^{\omega_i} \oplus n(\sin \lambda_j \ d_j)^{\omega_j}\right]^{\frac{2}{n(n+1)}} =$

$$\bigcup_{\gamma_i \epsilon h_i, \gamma_j \epsilon h_j, n_i \epsilon g_i, n_j \epsilon g_j} \left\{ \begin{array}{c} \left\{ \left(\sqrt{\prod_{i=1, j=i}^{n} [1 - [1 - (1 - \sin \lambda_i \sqrt{1-\gamma_i^2})^{\omega_i}]^m . [1 - (1 - \sin \lambda_j \sqrt{1-\gamma_j^2})^{\omega_j}]^n]^{\frac{2}{n(n+1)}}} \right) \right\}, \\ \left\{ \sqrt{1 - \prod_{i=1, j=i}^{n} [1 - \{1 - (1 - \sin \lambda_i \eta_i)^{\omega_i}\}^m . \{1 - (1 - \sin \lambda_j \eta_j)^{\omega_j}\}^n]^{\frac{2}{n(n+1)}}} \right\} \end{array} \right\}$$

THPFGHWG ($d_1, d_2 \ldots\ldots d_n$) $= \frac{1}{(m+n)} \left[\otimes_{i=1}^{n} \otimes_{j=1}^{n} m(\sin \lambda_i \ d_i)^{\omega_i} \oplus n(\sin \lambda_j \ d_j)^{\omega_j}\right]^{\frac{2}{n(n+1)}} =$

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\sqrt{1-\left[1-\prod_{i=1,j=i}^{n}\left\{1-\left[1-\left(1-\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\omega_i}\right]^m.\left[1-\left(1-\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\omega_j}\right]^n\right\}^{\frac{2}{n(n+1)}}\right]^{\frac{1}{(m+n)}}}\right\},\\ \left\{\left[\sqrt{1-\prod_{i=1,j=i}^{n}\left[1-\left\{1-(1-\sin\lambda_i\eta_i)^{\omega_i}\right\}^m.\left\{1-(1-\sin\lambda_j\eta_j)^{\omega_j}\right\}^n\right]^{\frac{2}{n(n+1)}}}\right]^{\frac{1}{(m+n)}}\right\}\end{array}\right\}$$

Consider the case of incomplete weight information where the criteria weights are in the form of intervals like $w_i = [a,b]$ where $a<b$ and $a, b \in [0,1]$. To deal with such situations the THPFGHWA and THPFGHWG are extended to handle incomplete weight information.

Definition 3.3.3. Let $d_i = (h_i, g_i)$ (i=1,2,3…n) be the collection of HPFNs and $0 \le \lambda_i \le 90$ then Trigonometric HPFN Generalized heronian interval weighted averaging operator THPFGHIWA: $\Omega^n \to \Omega$ is defined as

THPFGHIWA $(d_1, d_2 \ldots\ldots d_n)$ =

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\left[\sqrt{1-\prod_{i=1,j=i}^{n}\left[1-\left\{1-\left(\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\sqrt{b_i^2-a_i^2}}\right\}^m.\left\{1-\left(\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\sqrt{b_j^2-a_j^2}}\right\}^n\right]^{\frac{2}{n(n+1)}}}\right]^{\frac{1}{(m+n)}}\right\},\\ \left\{\sqrt{1-\left\{1-\prod_{i=1,j=i}^{n}\left[1-\left(1-(\sin\lambda_i\eta_i)^{\sqrt{b_i^2-a_i^2}}\right)^m.\left(1-(\sin\lambda_j\eta_j)^{\sqrt{b_j^2-a_j^2}}\right)^n\right]^{\frac{2}{n(n+1)}}\right\}^{\frac{1}{(m+n)}}}\right\}\end{array}\right\} \quad (19)$$

Here criteria weights are given as $w_i = [a_i, b_i]$ with $a_i<b_i$ and $a_i, b_i \in [0,1]$.

Definition 3.3.4. Let $d_i = (h_i, g_i)$ (i=1,2,3…n) be the collection of HPFNs and $0 \le \lambda_i \le 90$ then Trigonometric HPFN Generalized heronian interval weighted geometric operator THPFGHIWG: $\Omega^n \to \Omega$ is defined as

THPFGHIWG $(d_1, d_2 \ldots\ldots d_n)$ =

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\sqrt{1-\left[1-\prod_{i=1,j=i}^{n}\left\{1-\left[1-\left(1-\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\sqrt{b_i^2-a_i^2}}\right]^m.\left[1-\left(1-\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\sqrt{b_j^2-a_j^2}}\right]^n\right\}^{\frac{2}{n(n+1)}}\right]^{\frac{1}{(m+n)}}}\right\},\\ \left\{\left[\sqrt{1-\prod_{i=1,j=i}^{n}\left[1-\left\{1-(1-\sin\lambda_i\eta_i)^{\sqrt{b_i^2-a_i^2}}\right\}^m.\left\{1-(1-\sin\lambda_j\eta_j)^{\sqrt{b_j^2-a_j^2}}\right\}^n\right]^{\frac{2}{n(n+1)}}}\right]^{\frac{1}{(m+n)}}\right\}\end{array}\right\} \quad (20)$$

Here criteria weights are given as $w_i = [a_i, b_i]$ with $a_i<b_i$ and $a_i, b_i \in [0,1]$.

3.4. Weight determination method based on z-score

The weights of criteria have a big impact on the ranking of options throughout the MCDM process, thus choosing the proper ones is vital. A criterion with higher variance between alternatives has more weight, according to (Yingming 1997), and vice versa. As a result, this section presents a novel z-score-based weight determination approach. The z-score reflects how much standard deviations the value deviates from the mean. It aids in the establishment of modifications in criteria for each alternative and, as a result, in weighting. To decrease information loss, the choice matrix is turned into a membership and non-membership matrix.

Consider the Pythagorean fuzzy decision matrix as-

$$D = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array}\begin{bmatrix} C_1 & \cdots & C_n \\ \langle h_{11}, g_{11}\rangle & \cdots & \langle h_{1n}, g_{1n}\rangle \\ \vdots & \ddots & \vdots \\ \langle h_{m1}, g_{m1}\rangle & \cdots & \langle h_{mn}, g_{mn}\rangle \end{bmatrix}$$

Separate the matrix into membership matrix and non-membership matrix as-

$$M = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array}\begin{bmatrix} C_1 & \cdots & C_n \\ h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{m1} & \cdots & h_{mn} \end{bmatrix} \text{ and } N = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array}\begin{bmatrix} C_1 & \cdots & C_n \\ g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{m1} & \cdots & g_{mn} \end{bmatrix}$$

Find the mean and standard deviation of each column as-

$$\overline{u_j} = \frac{\sum_{i=1}^{m} u_{ij}}{m}\ , \quad \sigma_j = \sqrt{\frac{\sum_{i=1}^{m}(u_{ij}-\overline{u_j})^2}{m-1}}\ , \text{ i = 1, 2, ….m, j = 1, 2, …. n} \tag{21}$$

then compute the z-score of each value of column as:

$$z_{ij} = \frac{(u_{ij}-\bar{u}_j)}{\sigma_j}, \quad z_j = \sum_{i=1}^{m} |z_{ij}| \tag{22}$$

finally, the attribute weights are computed as:

$$\omega_j = \frac{z_j}{\sum_{j=1}^{n} z_j} \tag{23}$$

We get the weights of the membership matrix as $\omega = (\omega_1, \omega_2 \cdots \omega_n)^T$. Similarly, we can get the weights of non-membership matrix as $\overline{w} = (\overline{w}_1, \overline{w}_2 \cdots \overline{w}_n)^T$.

3.5. New Score function for hesitant Pythagorean fuzzy sets

Definition 3.5.1. (Garg 2018). Let $d_i = (h_i, g_i)$ be a HPFN, then the score function of d_i is defined as:

$$S(d_i) = \sum \gamma^2 / *hi \ - \ \sum \eta^2 / *gi \tag{24}$$

and the accuracy function as:

$$H(d_i) = \sum \gamma^2 / *hi \ + \ \sum \eta^2 / *gi \tag{25}$$

where $*hi$ and $*gi$ are the numbers of the elements in hi and gi, respectively.

Definition 3.5.2. (Khan et al. 2017). Let $d_i = (h_i, g_i)$ be a HPFN, then the score function of d_i is defined as:

$$S(d_i) = \left[\frac{\sum \gamma}{*hi}\right]^2 - \left[\frac{\sum \eta}{*gi}\right]^2 \tag{26}$$

and the accuracy function as:

$$H(d_i) = \left[\frac{\sum \gamma - S(di)}{*hi}\right]^2 - \left[\frac{\sum \eta - S(di)}{*gi}\right]^2 \tag{27}$$

where $*hi$ and $*gi$ are the numbers of the elements in hi and gi, respectively.

Example.3.5.1. Let $d_1 = \{\{0.2,0.3\}, \{0.5,0.6\}\}$ and $d_2 = \{\{0.1/\sqrt{2}, 0.5/\sqrt{2}\}, \{0.5,0.6\}\}$ be two HPFN then according to eq. (24)-

$S(d_1) = -0.24$ and $S(d_2) = -0.24$

On calculating the accuracy function $H(d_1) = H(d_2) = 0.37$

Hence the score and the accuracy function given by eq. (24) and (25) can't distinguish between d_1 and d_2.

Example.3.5.2. Let $d_1 = \{\{0.3,0.9\}, \{0.3,0.4,0.5\}\}$ and $d_2 = \{\{0.3,0.7,0.8\}, \{0.3,0.5\}\}$ be two HPFN then according to eq. (26)-

$S(d_1) = 0.2$ and $S(d_2) = 0.2$

On calculating the accuracy function $H(d_1) = H(d_2) = 0.2$

Hence the score and the accuracy function given by eq. (26) and (27) can't distinguish between d_1 and d_2.

Definition 3.5.3. Let $d_i = (h_i, g_i)$ be a HPFN, then the novel score function of d_i is defined as:

$$S(d_i) = \sum(1-\gamma)^2 \,/ *hi \;+\; \sum\eta^2 \,/ *gi \tag{28}$$

where $*hi$ and $*gi$ are the numbers of the elements in hi and gi, respectively.

The smaller the value of $S(d_i)$ the superior d_i is.

Example.3.5.3. Consider example 3.5.1 then according to the proposed score function-

$S(d_1) = 0.87$ and $S(d_2) = 0.9457$

Hence, $d_2 > d_1$.

Consider example 3.5.2 then according to the proposed score function-

$S(d_1) = 0.61667$ and $S(d_2) = 0.576667$

Hence, $d_1 > d_2$.

4. Application of the proposed approach

In this section we investigate the MAGDM problems based on the proposed approach.

4.1. Description of the problem

In today's highly competitive market, it is impossible for a business to make low-cost, high-quality things without the help of suppliers. When it comes to procuring the raw materials needed to support an organization's outputs, supplier selection is a common problem. Supplier evaluation and selection are critical choices for manufacturers' profitability, development, and survival in today's competitive global economy. Because they require the identification, study, and analysis of a vast number of physical factors, such judgements can be difficult to reach.

Consider a decision-making problem which consists of "m" different alternatives denoted by (A_1, A_2 …. A_m) that are evaluated under the set of "n" different criteria (C_1, C_2 …. Cn) by "k" decision makers (D_1, D_2 …. D_k). The weights of these criteria are completely unknown and the weight vector of decision makers be denoted by w = (w_1, w_2…. w_k) with $w_i, \omega_i \geq 0$ and $\sum_{i=1}^{k} w_i = 1, \sum_{i=1}^{n} \omega_i = 1$.

4.2. An algorithm to decision making approach

Let M_1, M_2 …. M_k be the normalized decision matrix given by DMs in which preference values are in the form of HPFNs. For normalization convert the beneficial criteria into non-beneficial or vice versa by taking complement of HPFNs.

$$M_1 = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{bmatrix} C_1 & \cdots & C_n \\ \langle h^1{}_{11}, g^1{}_{11} \rangle & \cdots & \langle h^1{}_{1n}, g^1{}_{1n} \rangle \\ \vdots & \ddots & \vdots \\ \langle h^1{}_{m1}, g^1{}_{m1} \rangle & \cdots & \langle h^1{}_{mn}, g^1{}_{mn} \rangle \end{bmatrix}$$

$$M_2 = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{bmatrix} C_1 & \cdots & C_n \\ \langle h^2{}_{11}, g^2{}_{11} \rangle & \cdots & \langle h^2{}_{1n}, g^2{}_{1n} \rangle \\ \vdots & \ddots & \vdots \\ \langle h^2{}_{m1}, g^2{}_{m1} \rangle & \cdots & \langle h^2{}_{mn}, g^2{}_{mn} \rangle \end{bmatrix}$$

$$M_k = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{bmatrix} C_1 & \cdots & C_n \\ \langle h^k{}_{11}, g^k{}_{11} \rangle & \cdots & \langle h^k{}_{1n}, g^k{}_{1n} \rangle \\ \vdots & \ddots & \vdots \\ \langle h^k{}_{m1}, g^k{}_{m1} \rangle & \cdots & \langle h^k{}_{mn}, g^k{}_{mn} \rangle \end{bmatrix}$$

where $\langle h^k{}_{ij}, g^k{}_{ij} \rangle$ are HPNs.

Step.1. Aggregate the values for each DM's by making the use of any above defined operators to form the aggregated matrix D.

$$D = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{bmatrix} C_1 & \cdots & C_n \\ \langle h_{11}, g_{11} \rangle & \cdots & \langle h_{1n}, g_{1n} \rangle \\ \vdots & \ddots & \vdots \\ \langle h_{m1}, g_{m1} \rangle & \cdots & \langle h_{mn}, g_{mn} \rangle \end{bmatrix}$$

Step.2. Convert the hesitant Pythagorean matrix into Pythagorean fuzzy matrix by taking the average of membership and non-membership degrees.

Step.3. Find the weights of the attributes by the z score method defined in eq. (23).

Step.4. Aggregate the values for each alternative by making use of above defined operators.

Step.5. Calculate the score value of each alternative using score function defined in eq. (28).

Step.6. Rank the alternatives in the ascending order of score values.

4.3. Numerical example

Consider a problem of supplier selection from four suppliers (Z1), (Z2), (Z3), (Z4) constructed on four criteria's quality (K1), price (K2), delivery time (K3), and the service (K4). The weights of three DM's are given as (0.3,0.2,0.5). Since criteria K2, K3 are non-beneficial and K1, K4 are beneficial. For normalization we convert beneficial criteria into non-beneficial by taking complement. The normalized decision matrix $D_i = (dij)4*4$ for all DM's is shown in Table 1, 2 and 3 where *dij* (i, j = 1,2,3,4) are from HPF environment.

Table 1
Hesitant Pythagorean fuzzy decision matrix for DM 1

	K_1	K_2	K_3	K_4
Z_1	{{0.3}, {0.6}}	{{0.4}, {0.4}}	{{0.2}, {0.7}}	{{0.1}, {0.3,0.4}}
Z_2	{{0.6}, {0.4}}	{{0.2}, {0.7}}	{{0.5}, {0.6,0.7}}	{{0.2,0.3}, {0.7}}
Z_3	{{0.5,0.7}, {0.2}}	{{0.3,0.5}, {0.8}}	{{0.3}, {0.4}}	{{0.1}, {0.8}}
Z_4	{{0.7}, {0.3}}	{{0.4}, {0.6}}	{{0.1}, {0.6}}	{{0.3}, {0.7}}

Table 2
Hesitant Pythagorean fuzzy decision matrix for DM 2

	K_1	K_2	K_3	K_4
Z_1	{{0.1}, {0.8}}	{{0.5}, {0.6,0.7}}	{{0.3}, {0.7,0.8}}	{{0.3}, {0.6}}
Z_2	{{0.7}, {0.3}}	{{0.2}, {0.7}}	{{0.6}, {0.4}}	{{0.3,0.6}, {0.4}}
Z_3	{{0.3,0.5}, {0.8}}	{{0.1}, {0.6}}	{{0.1}, {0.5}}	{{0.1}, {0.4}}
Z_4	{{0.6}, {0.4}}	{{0.1}, {0.8}}	{{0.2,0.4}, {0.6}}	{{0.7}, {0.3}}

Table 3
Hesitant Pythagorean fuzzy decision matrix for DM 3

	K_1	K_2	K_3	K_4
Z_1	{{0.5,0.7}, {0.2}}	{{0.6}, {0.4}}	{{0.2}, {0.7,0.8}}	{{0.4}, {0.5}}
Z_2	{{0.1}, {0.7}}	{{0.6}, {0.3}}	{{0.5}, {0.4,0.5}}	{{0.6}, {0.2}}
Z_3	{{0.2}, {0.6}}	{{0.6,0.8}, {0.2}}	{{0.1}, {0.6}}	{{0.5}, {0.3,0.6}}
Z_4	{{0.4,0.5}, {0.5}}	{{0.1}, {0.8}}	{{0.7}, {0.3}}	{{0.2,0.4}, {0.4}}

Step.1. Aggregate the values for each DM's by making the use of THPFWA operator with weights of DM's as (0.3,0.2,0.5). The aggregated matrix is shown in table 4.

Table 4
Aggregate decision matrix

	K_1	K_2	K_3	K_4
Z_1	{{0.734,0.761}, {0.436}}	{{0.755}, {0.474,0.481}}	{{0.715}, {0.598,0.617,0.606,0.625}}	{{0.724}, {0.48,0.5}}
Z_2	{{0.748}, {0.507}}	{{0.743}, {0.486}}	{{0.754}, {0.483,0.511,0.494,0.522}}	{{0.745,0.754,0.747,0.756}, {0.416}}
Z_3	{{0.726,0.743,0.732,0.749}, {0.485}}	{{0.745,0.782,0.753,0.789}, {0.442}}	{{0.713}, {0.514}}	{{0.730}, {0.469,0.55}}
Z_4	{{0.761,0.769}, {0.461}}	{{0.716}, {0.612}}	{{0.758,0.762}, {0.468}}	{{0.735,0.744}, {0.48}}

Step.2. Convert the hesitant Pythagorean fuzzy matrix into Pythagorean fuzzy matrix for calculating weights.
The converted matrix is shown in Table 5.

Table 5
Pythagorean fuzzy decision matrix

	K_1	K_2	K_3	K_4
Z_1	{{0.747}, {0.436}}	{{0.755}, {0.477}}	{{0.715}, {0.611}}	{{0.724}, {0.49}}
Z_2	{{0.748}, {0.507}}	{{0.743}, {0.486}}	{{0.754}, {0.502}}	{{0.751}, {0.416}}
Z_3	{{0.737}, {0.485}}	{{0.767}, {0.442q}}	{{0.713}, {0.514}}	{{0.73}, {0.509}}
Z_4	{{0.765}, {0.461}}	{{0.716}, {0.612}}	{{0.76}, {0.468}}	{{0.739}, {0.48}}

Step.3. By applying the z-score method as described in section 3.4 we get the weights of membership and non-membership matrix as:
$\omega = (0.224, 0.238, 0.284, 0.254)^T$ and $\bar{w} = (0.265, 0.248, 0.243, 0.244)^T$.

Step.4. By employing the THPFWA operator in Eq. (7) to Table 4, we get the overall value of the alternatives *Zi* (*i* = 1,2,3,4). We let $\lambda_1 = 10$, $\lambda_2 = 30$, $\lambda_3 = 50$, $\lambda_4 = 70$. For instance, the overall value of *Z*1 is-

$z_1 = \text{THPFWA}(z_{11}, z_{12}, z_{13}, z_{14})$

$$= \bigcup_{\gamma_{1j}\epsilon h_{1j}, \eta_{1j}\epsilon g_{1j},} \left\{ \left\{ \sqrt{1 - \prod_{j=1}^{4} \left[\sin \lambda_j \sqrt{1 - \gamma_{1j}{}^2}\right]^{\omega_j}} \right\}, \left\{ \prod_{j=1}^{4} \sqrt{(\sin \lambda_j \eta_{1j})}^{\bar{w}_j} \right\} \right\}$$

= {{0.78804, 0.790479}, {0.510428, 0.511337, 0.512664, 0.513578, 0.512186, 0.513098, 0.51443, 0.515346, 0.511176, 0.512086, 0.513415, 0.51433, 0.512908, 0.513822, 0.515155, 0.516073}}

Equally for other alternatives,

z_2 = {{0.793376, 0.794079, 0.793529, 0.79424}, {0.50197, 0.5052, 0.503261, 0.506419}

z_3 = {{0.786676, 0.790402, 0.787429, 0.791183, 0.788135, 0.791832, 0.788883, 0.792607, 0.787179, 0.790896, 0.787931, 0.791675, 0.788675, 0.792362, 0.789421, 0.793134}, {0.503406, 0.511969}

z_4 = {{0.79199, 0.792763, 0.792417, 0.793189, 0.792659, 0.793429, 0.793085, 0.793854}, {0.508902}}

Step.5 Calculate the score $S(zi)$ of the overall HPFE di (i = 1, 2, 3, 4) by using eq. (28).
$S(z1) = 0.307843$, $S(z2) = 0.296749$, $S(z3) = 0.301911$, $S(z4) = 0.301863$

Step.6 Rank the alternatives in the ascending order of scores.

$Z_2 > Z_4 > Z_3 > Z_1$. Hence the most required alternative is Z_2.

Similarly, on applying THPFWG operator to aggregate the values we get the following results. The first three steps are same.

Step.1' By utilizing the THPFWG operator in eq. (11) we get the overall value of the alternatives Zi (i = 1,2,3,4). We let $\lambda_1 = 10$, $\lambda_2 = 30$, $\lambda_3 = 50$, $\lambda_4 = 70$. For example, overall value of Z_1 is:

z_1 = THPFWG ($d_{11}, d_{12}, d_{13}, d_{14}$)

$$= \bigcup_{\gamma_{1j}\epsilon h_{1j}, n_{1j}\epsilon g_{1j}} \left\{ \left\{ \prod_{j=1}^{4} \left\{ \sqrt{1-\left[\sin\lambda_j \sqrt{1-\gamma_{1j}^2}\right]} \right\}^{\omega_j} \right\}, \left\{ \sqrt{1-\prod_{j=1}^{4}(1-\sin\lambda_j \eta_{1j})^{\overline{w}_j}} \right\} \right\}$$

= {{0.69923, 0.699697}, {0.604582, 0.605195, 0.610436, 0.611037, 0.608244, 0.608849, 0.614024, 0.614616, 0.606121, 0.606731, 0.611944, 0.612541, 0.609793, 0.610395, 0.615541, 0.61613}}

Equally for other alternatives,

z_2 = {{0.71194, 0.714753, 0.712562, 0.715383}, {0.566335, 0.571852, 0.568497, 0.574031}}

z_3 = {{0.699853, 0.702551, 0.700415, 0.703092, 0.700136, 0.702835, 0.700698, 0.703376, 0.699952, 0.70265, 0.700513, 0.703191, 0.70024, 0.702939, 0.700802, 0.70348}, {0.58367, 0.609078}}

z_4 = {{0.708001, 0.708148, 0.708748, 0.708896, 0.710769, 0.710917, 0.71152, 0.711667}, {0.59291}}

Step.2' Calculate the score $S(zi)$ of the overall HPFE di (i = 1, 2, 3, 4) using eq. (28).

$S(z1) = 0.462906$, $S(z2) = 0.407106$, $S(z3) = 0.444826$, $S(z4) = 0.435743$

Step.3' Rank the alternatives in the ascending order of scores.

$Z_2 > Z_4 > Z_3 > Z_1$. Hence the most desirable alternative is Z_2.

Similarly, we apply THPFOWA, THPFOWG operators to get the ranking.

On applying THPFGHWA operator to aggregate the values we get the following results.

Step.1. We let $\lambda_1 = 10$, $\lambda_2 = 30$, $\lambda_3 = 50$, $\lambda_4 = 70$ and m = n = 2. For instance, for alternative Z1 we have-

z_1 = THPFGHWA ($z_{11}, z_{12}, z_{13}, z_{14}$)

$$= \bigcup_{\gamma_i\epsilon h_i, \gamma_j\epsilon h_j, n_i\epsilon g_i, n_j\epsilon g_j} \left\{ \begin{array}{c} \left\{ \left[\sqrt{1-\prod_{i=1,j=i}^{4}\left[1-\left\{1-\left(\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\omega_i}\right\}.\left\{1-\left(\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\omega_j}\right\}\right]^{\frac{1}{10}}} \right]^{\frac{1}{(2)}} \right\}, \\ \left\{ \sqrt{1-\left\{1-\prod_{i=1,j=i}^{4}\left[1-\left(1-(\sin\lambda_i\eta_i)^{\overline{w}_i}\right).\left(1-(\sin\lambda_j\eta_j)^{\overline{w}_j}\right)\right]^{\frac{1}{10}}\right\}^{\frac{1}{(2)}}} \right\} \end{array} \right\}$$

= {{0.782917, 0.7844307}, {0.824263, 0.824576, 0.82468, 0.824931, 0.824586, 0.8249, 0.824941, 0.825255, 0.824402, 0.824715, 0.824757, 0.82507, 0.824715, 0.825029, 0.82507, 0.825384}}

Likewise for other alternatives,

z_2 = {{0.786539, 0.787044, 0.786649, 0.78716}, {0.826621, 0.827387, 0.826932, 0.827665}}

z_3 = {{0.782092, 0.784548, 0.782589, 0.785061, 0.782923, 0.785367, 0.783418, 0.785878, 0.782379, 0.784831, 0.782875, 0.785343, 0.783231, 0.78567, 0.783725, 0.78618}}

z_4 = {{0.78547, 0.78591, 0.785767, 0.786206, 0.78595, 0.786389, 0.786246, 0.786684}, {0.824436}}

Step.2. Calculate the score $S(zi)$ of the overall HPFE di (i = 1, 2, 3, 4).

$S(z1) = 0.727162$, $S(z2) = 0.729614$, $S(z3) = 0.728121$, $S(z4) = 0.725457$

Step.3. Rank the alternatives in the ascending order of scores.

$Z_4 > Z_1 > Z_3 > Z_2$. Hence the most desirable alternative is Z_4.

Similarly, by using THPFGHWG operator to aggregate the values we get the following results.

Step.1' We let $\lambda_1 = 10$, $\lambda_2 = 30$, $\lambda_3 = 50$, $\lambda_4 = 70$ and m = n = 2. For instance, for alternative Z1 we have-

z_1 = THPFGHWG (z_{11}, z_{12}, z_{13}, z_{14})

$$\bigcup_{\gamma_i\epsilon h_i,\gamma_j\epsilon h_j,\eta_i\epsilon g_i,\eta_j\epsilon g_j}\left\{\begin{array}{c}\left\{\left(\sqrt{1-\left[1-\prod_{i=1,j=i}^{4}\left\{1-\left[1-\left(1-\sin\lambda_i\sqrt{1-\gamma_i^2}\right)^{\omega_i}\right]^2.\left[1-\left(1-\sin\lambda_j\sqrt{1-\gamma_j^2}\right)^{\omega_j}\right]^2\right\}\right]^{\frac{1}{10}}}\right)^{\frac{1}{4}}\right\},\\ \left\{\left(\left[\sqrt{1-\prod_{i=1,j=i}^{4}\left[1-\left\{1-(1-\sin\lambda_i\eta_i)^{\varpi_i}\right\}^2.\left\{1-(1-\sin\lambda_j\eta_j)^{\varpi_j}\right\}^2\right]}\right]^{\frac{1}{10}}\right)^{\frac{1}{4}}\right\}\end{array}\right\}$$

= {{0.896328, 0.896346}, {0.357018, 0.357181, 0.362214, 0.362369, 0.359667, 0.359826, 0.364706, 0.364858, 0.35812, 0.358282, 0.36325, 0.363404, 0.360814, 0.360972, 0.365787, 0.365938}}

Likewise for other alternatives,

z_2 = {{0.901686, 0.903092, 0.901998, 0.903405}, {0.325612, 0.32928, 0.327021, 0.330795}}

z_3 = {{0.89701, 0.897366, 0.897087, 0.897433, 0.897022, 0.897377, 0.897098, 0.897445, 0.897014, 0.89737, 0.897091, 0.897437, 0.897026, 0.897381, 0.897102, 0.897449}, {0.342816, 0.367274}}

z_4 = {{0.900112, 0.900117, 0.900343, 0.900349, 0.90151, 0.901516, 0.901746, 0.901752}, {0.346257}}

Step.2' Calculate the score $S(zi)$ of the overall HPFE di (i = 1, 2, 3, 4).

$S(z1) = 0.141455$, $S(z2) = 0.117202$, $S(z3) = 0.136768$, $S(z4) = 0.129709$

Step.3' Rank the alternatives in the ascending order of scores.

$Z_2 > Z_4 > Z_3 > Z_1$. Hence the most desirable alternative is Z_2.

Considering real life problems there may arise situations when the information about attribute weights is incomplete i.e., despite of a particular crisp value of weight it lies between particular interval such as $w_i = [a, b]$ where $a<b$ and a, b Є [0,1]. In such cases we can use THPFGHIWA and THPFGHIWG operators.

Considering the same problem let the criteria weights be given as:
w_1 Є [0.2, 0.3], w_2 Є [0.18, 0.27], w_3 Є [0.25, 0.35] and w_4 Є [0.15, 0.25]

Applying THPFGHIWG operator to aggregate the values we get the following results.

Step.1 By utilizing the THPFGHIWG operator, we get the overall value of the alternatives Zi (i = 1,2,3,4).
Let $\lambda_1 = 10$, $\lambda_2 = 30$, $\lambda_3 = 50$, $\lambda_4 = 70$ and m = n = 2. For instance, for an alternative Z1 we have-

z_1 = THPFGHIWG (d_{11}, d_{12}, d_{13}, d_{14})

$$\bigcup_{\gamma_i \epsilon h_i, \gamma_j \epsilon h_j, n_i \epsilon g_i, n_j \epsilon g_j} \left\{ \begin{array}{c} \left\{ \left\{ \sqrt{1 - \left[1 - \prod_{i=1,j=i}^{4} \left\{ 1 - \left[1 - \left(1 - \sin\lambda_i \sqrt{1-\gamma_i^2}\right)^{\sqrt{b_i^2 - a_i^2}} \right]^2 . \left[1 - \left(1 - \sin\lambda_j \sqrt{1-\gamma_j^2}\right)^{\sqrt{b_j^2 - a_j^2}} \right]^2 \right\}^{\frac{2}{10}} \right]^{\frac{1}{(4)}}} \right\} \right\}, \\ \left\{ \left[\sqrt{1 - \prod_{i=1,j=i}^{4} \left[1 - \left\{ 1 - (1 - \sin\lambda_i \eta_i)^{\sqrt{b_i^2 - a_i^2}} \right\}^2 . \left\{ 1 - (1 - \sin\lambda_j \eta_j)^{\sqrt{b_j^2 - a_j^2}} \right\}^2 \right]^{\frac{2}{10}}} \right]^{\frac{1}{(4)}} \right\} \end{array} \right\}$$

= {{0.914243, 0.914265}, {0.355484, 0.355952, 0.360752, 0.360854, 0.358191, 0.358296, 0.363296, 0.363396, 0.35661, 0.356717, 0.36181, 0.361911, 0.359364, 0.359468, 0.3644, 0.364499}}

Likewise for other alternatives,

z_2 = {{0.918829, 0.919949, 0.919078, 0.920198}, {0.323165, 0.326953, 0.32462, 0.328516}}

z_3 = {{0.91476, 0.91507, 0.91483, 0.91512, 0.91477, 0.91508, 0.91484, 0.91514, 0.91476, 0.91507, 0.91483, 0.91513, 0.91478, 0.91508, 0.91484, 0.91517}, {0.31244, 0.366021}}

z_4 = {{0.917587, 0.917594, 0.9178804, 0.917811, 0.918706, 0.918713, 0.918927, 0.918934}, {0.342882}}

Step.2 Calculate the score $S(zi)$ of the overall HPFE di (i = 1, 2, 3, 4).

$S(z1) = 0.13699$, $S(z2) = 0.112637$, $S(z3) = 0.132435$, $S(z4) = 0.12425$

Step.3 Rank the alternatives in the ascending order of scores.

$Z_2 > Z_4 > Z_3 > Z_1$. Hence the most desirable alternative is Z_2.

Similarly, we can use THPFGHIWA operator to aggregate the values, the ranking order thus obtained is

$Z_4 > Z_1 > Z_2 > Z_3$. Hence the most desirable supplier is Z_4.

Figure 2 depicts the ranking order by different operators.

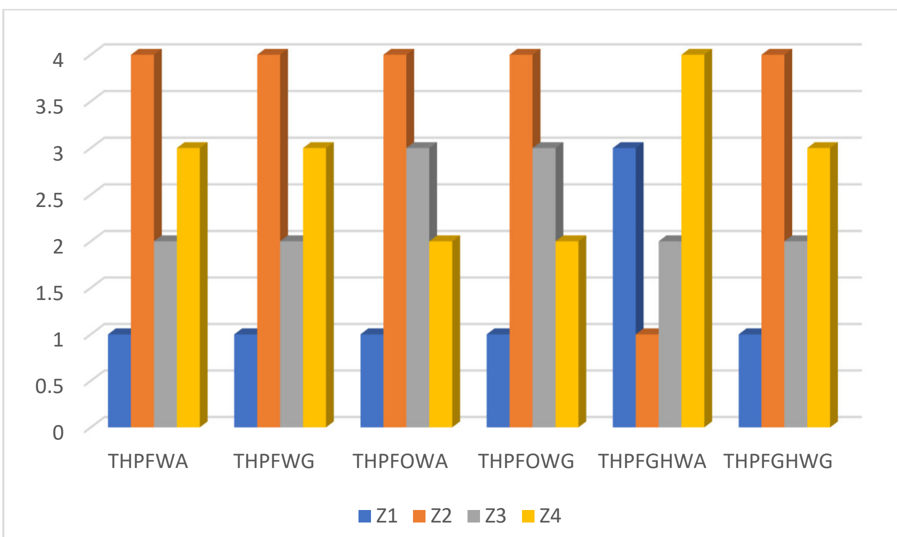

Fig. 2. Graphical representation of the ranking by different operators

5. Comparative analysis and discussions

In this section, we illustrate the validity of the proposed approach by comparing with many other well-known approaches and discuss the results.

5.1. *Comparative analysis*

Here we compare the ranking obtained by the described operators with the other well-known aggregation's operators for HPFNs. Consider the numerical example given in section 4.3.

Table 6
Comparative Study

Aggregation Operators	Ranking of alternatives
HPFWA	$Z_2 > Z_4 > Z_3 > Z_1$
HPFWG	$Z_2 > Z_4 > Z_3 > Z_1$
HPFOWA	$Z_2 > Z_3 > Z_4 > Z_1$
HPFOWG	$Z_2 > Z_3 > Z_4 > Z_1$
DHPFWA	$Z_2 > Z_4 > Z_3 > Z_1$
DHPFWG	$Z_2 > Z_3 > Z_4 > Z_1$
DHPFGWHM	$Z_2 > Z_4 > Z_3 > Z_1$
DHPFGGWHM	$Z_2 > Z_4 > Z_3 > Z_1$
THPFWA	$Z_2 > Z_4 > Z_3 > Z_1$
THPFWG	$Z_2 > Z_4 > Z_3 > Z_1$
THPFOWA	$Z_2 > Z_3 > Z_4 > Z_1$
THPFOWG	$Z_2 > Z_3 > Z_4 > Z_1$
THPFGHWA	$Z_4 > Z_1 > Z_3 > Z_2$
THPFGHWG	$Z_2 > Z_4 > Z_3 > Z_1$

We can also implement fuzzy TOPSIS in HPF environment to rank the alternatives, the closeness coefficient thus obtained are CC_1= 0.7255, CC_2= 0.4738, CC_3= 0.3177, CC_4=0.7505. The ranking of alternatives is given as $Z_4 > Z_1 > Z_2 > Z_3$. The difference in ranking is because of the fact that the proposed method effectively considers the interaction between membership and non-membership degree which is absent in TOPSIS method. Also, the study assigns attribute weights for membership and non-membership degrees separately to reduce the information loss, thus provides more reliable results as compared to TOPSIS in HPF environment.

Furthermore, it is observed from Table 6 that the best alternative obtained by our proposed operators coincide with the existing operators under hesitant Pythagorean fuzzy environment. Hence this confirms the resistance of the proposed operators. From the table we can verify that the optimal alternative is A_2 by all the approaches, the computational steps being different. The proposed operators use the characteristics of sine trigonometric function to aggregate the hesitant Pythagorean fuzzy information hence provides effective results as compared to the existing ones. Also, the consideration of optimistic nature of decision-makers makes the results more practical. Thus, the proposed method plays a vital role in solving real decision-making problems.

5.2. Influence of value of parameter λ

With the help of trigonometric properties and definition of $\sin_{\lambda} d$ we observe that-

a) There exist a $\lambda^* = \sin^{-1}\sqrt{1-\gamma^2}$ such that
   - If $\lambda = \lambda^*$, then $\sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}} = \gamma$
   - If $\lambda > \lambda^*$, then $\sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}} < \gamma$
   - If $\lambda < \lambda^*$, then $\sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}} > \gamma$

b) There exist a $\lambda^{\#} = \sin^{-1}\eta$ such that
   - If $\lambda = \lambda^{\#}$, then $\sqrt{\sin\eta} = \eta$
   - If $\lambda > \lambda^{\#}$, then $\sqrt{\sin\eta} > \eta$
   - If $\lambda < \lambda^{\#}$, then $\sqrt{\sin\eta} < \eta$

b) If we pick $\lambda < \lambda^* < \lambda^{\#}$, then $\sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}} > \gamma$ and $\sqrt{\sin\eta} < \eta$ thus $\sin_{\lambda} d > d$. Hence on subsequently applying the trigonometric operator, the estimation of HPFN d will be expanded.

d) If we pick $\lambda^* < \lambda < \lambda^{\#}$, then $\sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}} < \gamma$ and $\sqrt{\sin\eta} < \eta$ thus values of both membership degree and non-membership degree decrease on applying trigonometric operator.

e) If we pick $\lambda^* < \lambda^{\#} < \lambda$, then $\sqrt{1-\sin\lambda\sqrt{(1-\gamma^2)}} < \gamma$ and $\sqrt{\sin\eta} > \eta$ thus values of membership degrees decrease and non-membership degree increases on applying trigonometric operator.

Thus, keeping all these points in mind, a decision maker can choose a suitable value of parameter λ according to their risk preference.

5.3. Discussions

The main difference between the proposed approach and existing methods is shown in Table 7.

Table 7
Comparative analysis

Aggregation operators	Whether attribute weights are derived using some information measures	Whether consider interrelationship among arguments	Whether contains an additional parameter	Whether information by HPFS	Whether can handle GDM problems	Whether define score function	Whether can handle incomplete weights
(Garg 2018)	No	No	No	Yes	No	Yes	No
(Tang et al. 2019)	No	Yes	No	Yes	No	No	No
(Yang et al. 2019)	No	Yes	No	Yes	No	No	No
(Rahman et al. 2020)	No	No	Yes	No	Yes	No	No
(Sarkar et al. 2021)	No	Yes	No	Yes	No	No	No
(Senapati et al. 2022)	No	No	Yes	No	No	No	No
Proposed approach	Yes	Yes	Yes	Yes	Yes	Yes	Yes

To highlight the superiority of the proposed approach different characteristics of the proposed approach are discussed over the existing ones. From the table we can determine that the approaches mentioned in (Rahman et al., 2020) and (Senapati et al. 2022) are bound to deliver their preferences in form of discrete set (PFS) and hence the approaches are precarious. The proposed approach considers more general case of decision-making in real environment where decision makers preferences consider hesitancy. The existing approaches except (Rahman et al., 2020) are applied for solving MADM problems and doesn't consider GDM problems, but GDM provides more realistic results as it considers the opinions of different experts. The attribute weights in the existing approaches are taken as priori which is quite inappropriate. Due to lack of complete information, it is unlikely to get weight information before deciding alternatives. In real-life MADM problems relationship do exist between the arguments being fused. Thus, the interrelationship among arguments is an important aspect which can't be ignored in decision-making process. (Garg et al., 2018; Rahman et al. 2020, Senapati et al. 2022) doesn't take the relationship among arguments into consideration. Risk preferences of decision makers can alter the final ranking of alternatives; thus, it becomes essential to capture the risk preference of decision makers before final ranking. The proposed approach considers an additional parameter to capture the risk preference of DMs, the DMs makers can vary the value of the parameters according to their tendencies, thus the operators with the parameter provides more versatile and realistic results as compared to the approaches in (Garg et al., 2018; Tang et al., 2019; Yang et al., 2019; Sarkar et al., 2021). Score function provides score value to an alternative, responsible for the ranking order of alternatives. But the score functions defined in (Khan et al., 2017; Garg 2018) are not able to provide ranking order in some situations thus, the proposed approach conquers the shortcomings of existing score functions to provide more realistic outcomes. Also, it is not always possible for the experts to have complete information about attribute weights, thus the study extends the defined operators to the environment of incomplete weight information, to robust the results as compared to the other operators defined in the existing approaches.

However, the present study considers operational laws based on sine trigonometric function to handle the ambiguity in data, responsible for information loss through analysis. Also, the proposed approach is capable of handling GDM problems. Besides, the attribute weights are not known beforehand and are deduced based on the concept of z-scores considering the membership and non-membership degrees separately to reduce the original information loss. Further, the operators consider the relationship between arguments and risk preference of decision makers. Thus, the proposed operators are more comprehensive, reliable and contains broader information to handle GDM problems in HPF environment.

6. Experimental evaluations

In this section we verify the stability of the proposed approach by carrying sensitivity analysis.

There are two parameters namely m, n by making the use of generalized Heronian mean operator for taking the relationship between arguments into account. By varying the values of these parameters, we can get different rankings. Earlier also in many studies researchers have used operators like generalized heronian mean operators, generalized bonferroni mean operators etc. In the present study in addition to these parameters there exists an extra parameter λ which makes it more flexible. As for a particular value of m, n different values of λ can be varied according to the risk preference of decision makers. Also, from Table 8 it is concluded that the best alternative will remain same in all the cases hence there will be no conflict in selecting the best alternative.

Table 8
Ranking by THPFGHWG

m = n = 4	
$\lambda_1 = 05, \lambda_2 = 35, \lambda_3 = 45, \lambda_4 = 85$	$Z_2 > Z_4 > Z_1 > Z_3$
$\lambda_1 = 10, \lambda_2 = 20, \lambda_3 = 80, \lambda_4 = 90$	$Z_2 > Z_4 > Z_3 > Z_1$
m = n = 6	
$\lambda_1 = 08, \lambda_2 = 15, \lambda_3 = 85, \lambda_4 = 90$	$Z_2 > Z_4 > Z_3 > Z_1$
$\lambda_1 = 05, \lambda_2 = 45, \lambda_3 = 45, \lambda_4 = 90$	$Z_2 > Z_4 > Z_1 > Z_3$

7. Conclusion

The major contributions of the present study are described as:

1. Hesitant Pythagorean fuzzy sets, an extension of PFS and HFS, can handle indeterminacy in a far better way, thus the paper makes the use of HPFS to deal with the uncertainty persisting in the opinion of decision makers.
2. The paper aims to define aggregation operators in HPF environment. Since during the aggregation, the crucial process is to outline operational laws which provides a base for aggregation operators this paper defines some novel trigonometric operational laws for HPFNs by making the use of sine function keeping in mind its characteristics such as periodicity, symmetry about origin and restricted range. Fundamental properties of these operational laws are studied and discussed in detail.
3. The flexibility of the proposed operational laws is increased by considering an additional parameter λ to measure the preferences of the decision makers. The decision makers can vary the value of the parameter according to their risk preference as shown in section 5.2.
4. To aggregate the information in HPF environment, different aggregation operators are defined built on these operational laws. Several properties related to the proposed operators are discussed.
5. To strengthen the operators, they are combined with generalized heronian mean operator which measures the relationship between arguments to be aggregated.
6. Since ranking of alternatives is affected by weights of criteria, and it is unlikely to always have a prior information about attribute weights due to lack of information, thus the study defines a novel weight determination method based on the concept of z-scores. The weight of criterions is determined by taking the membership and non-membership degrees separately to reduce the loss in original information.
7. Considering the real-life problems there might arise the situation when DMs are not able to assign an exact weight to attribute rather, they define an interval in which weights may lie. Thus, the defined operators are extended to the environment with incomplete weight information.
8. Further a score function for HPFN's is defined which conquers the shortcomings of previously defined score functions (Khan et al. 2017, Garg 2018) in HPF environment.
9. A complete algorithm for multi attribute group decision-making is presented based on the defined aggregation operators where each alternative is evaluated by different decision makers in hesitant Pythagorean fuzzy environment.
10. The effectiveness and reliability of the proposed approach is investigated with the help of numerical example of supplier selection problem and compared with the existing approaches in HPF environment. The discussion in Table 7 explains the superiority of the proposed approach. The robustness of the proposed approach is enhanced using sensitivity analysis in Table 8.

Analyzing the abovementioned points, it can be concluded that the proposed approach and operators can effectively handle the group decision-making problems in HPF environment.

In the future, these operators can be employed to solve decision-making problems in other fields such as medical diagnosis, transportation problems etc. Further, these operators can be extended to other variants of fuzzy sets like neutrosophic fuzzy sets etc.

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Appendix A. List of abbreviations used

MAGDM	Multi-Attribute Group Decision-Making
HPFS	Hesitant Pythagorean Fuzzy Sets
HPFE	Hesitant Pythagorean Fuzzy Elements
HM	Heronian Mean operator
GHM	Generalized Heronian Mean operator
TOLs	Trigonometric Operational Laws
THPFWA	Trigonometric Hesitant Pythagorean Fuzzy Weighted Averaging operator
THPFOWA	Trigonometric Hesitant Pythagorean Fuzzy Ordered Weighted Averaging operator
THPFWG	Trigonometric Hesitant Pythagorean Fuzzy Weighted Geometric operator
THPFOWG	Trigonometric Hesitant Pythagorean Fuzzy Ordered Weighted Geometric operator
THPFGHWA	Trigonometric Hesitant Pythagorean Fuzzy Generalized Heronian Weighted Averaging operator
THPFGHWG	Trigonometric Hesitant Pythagorean Fuzzy Generalized Heronian Weighted Geometric operator
THPFGHIWA	Trigonometric Hesitant Pythagorean Fuzzy Generalized Heronian Interval Weighted Averaging operator
THPFGHIWG	Trigonometric Hesitant Pythagorean Fuzzy Generalized Heronian Interval Weighted Geometric operator

Appendix B. Proof of theorem 3.1.2

$$\sin_\lambda d_1 = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \sqrt{1-\sin\lambda\sqrt{(1-\gamma_1{}^2)}}, \sqrt{\sin\lambda\eta_1} \right\}$$

$$\sin_\lambda d_2 = \bigcup_{\gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \sqrt{1-\sin\lambda\sqrt{(1-\gamma_2{}^2)}}, \sqrt{\sin\lambda\eta_2} \right\}$$

By using operational laws for HPFNs, we have

$$\sin_\lambda d_1 \oplus \sin_\lambda d_2 = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \sqrt{1-\sin\lambda\sqrt{(1-\gamma_1{}^2)}\,.\sin\lambda\sqrt{(1-\gamma_2{}^2)}}\ ,\ \sqrt{\sin\lambda\eta_1 \sin\lambda\eta_2} \right\}$$

$$\sin_\lambda d_1 \otimes \sin_\lambda d_2 = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \sqrt{(1-\sin\lambda\sqrt{(1-\gamma_1{}^2)}\,)\,.\,(1-\sin\lambda\sqrt{(1-\gamma_2{}^2)})}\ , \sqrt{1-(1-\sin\lambda\eta_1\,).(1-\sin\lambda\eta_2)} \right\}$$

(a) for a real number $k > 0$, $k\,(\sin_\lambda d_1 \oplus \sin_\lambda d_2)$

$$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \sqrt{(1-(\sin\lambda\sqrt{(1-\gamma_1{}^2)}\,.\sin\lambda\sqrt{(1-\gamma_2{}^2)})^k}\ , \sqrt{(\sin\lambda\eta_1 \sin\lambda\eta_2)}^k \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \sqrt{(\sin\lambda\sqrt{(1-\gamma_1{}^2)}\,)^k}, \sqrt{\sin\lambda\eta_1{}^k} \right\} \oplus \bigcup_{\gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \sqrt{(\sin\lambda\sqrt{(1-\gamma_2{}^2)}\,)^k}, \sqrt{\sin\lambda\eta_2{}^k} \right\}$$

$$= k \sin_\lambda d_1 \oplus k \sin_\lambda d_2$$

(b) for a real number $k > 0$, $(\sin_\lambda d_1 \otimes \sin_\lambda d_2)^k$

$$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \sqrt{(1-\sin\lambda\sqrt{(1-\gamma_1{}^2)})^k.\,(1-\sin\lambda\sqrt{(1-\gamma_2{}^2)})^k}\ , \sqrt{1-((1-\sin\lambda\eta_1)(1-\sin\lambda\eta_2))^k} \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \left[\sqrt{1-\sin\lambda\sqrt{(1-\gamma_1{}^2)}}\right]^k, \sqrt{1-(1-\sin\lambda\eta_1)^k} \right\} \otimes \bigcup_{\gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \left[\sqrt{1-\sin\lambda\sqrt{(1-\gamma_2{}^2)}}\right]^k, \sqrt{1-(1-\sin\lambda\eta_2)^k} \right\}$$

$$= \sin_\lambda d_1{}^k \otimes \sin_\lambda d_2{}^k$$

(c) for a real number $k_1, k_2 > 0$, $(k_1 \sin_\lambda d_1 \oplus k_2 \sin_\lambda d_1)$

$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \sqrt{(\sin \lambda \sqrt{(1-\gamma_1{}^2)}\,)^{k1}}, \sqrt{\sin \lambda \eta_1{}^{k1}} \right\} \oplus$

$\bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \sqrt{(\sin \lambda \sqrt{(1-\gamma_1{}^2)}\,)^{k2}}, \sqrt{\sin \lambda \eta_1{}^{k2}} \right\}$

$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \sqrt{(\sin \lambda \sqrt{(1-\gamma_1{}^2)}\,)^{k1+k2}}, \sqrt{\sin \lambda \eta_1{}^{k1+k2}} \right\}$

$= (k1 + k2)\sin_\lambda d_1$

(d) for a real number $k_1, k_2 > 0$, $(\sin_\lambda d_1)^{k_1} \otimes (\sin_\lambda d_1)^{k_2}$

$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \left[\sqrt{1-\sin \lambda \sqrt{(1-\gamma_1{}^2)}}\right]^{k1}, \sqrt{1-(1-\sin \lambda \eta_1)^{k1}} \right\} \otimes$

$\bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \left[\sqrt{1-\sin \lambda \sqrt{(1-\gamma_1{}^2)}}\right]^{k2}, \sqrt{1-(1-\sin \lambda \eta_1)^{k2}} \right\}$

$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \left[\sqrt{1-\sin \lambda \sqrt{(1-\gamma_1{}^2)}}\right]^{k1+k2}, \sqrt{1-(1-\sin \lambda \eta_1)^{k1+k2}} \right\}$

$= (\sin_\lambda d_1)^{k_1+k_2}$

(e) for a real number $k_1, k_2 > 0$, $((\sin_\lambda d_1)^{k_1})^{k_2}$

$= \langle \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \left[\sqrt{1-\sin \lambda \sqrt{(1-\gamma_1{}^2)}}\right]^{k1}, \sqrt{1-(1-\sin \lambda \eta_1)^{k1}} \right\} \rangle^{k2}$

$= \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \left[\sqrt{1-\sin \lambda \sqrt{(1-\gamma_1{}^2)}}\right]^{k1 \cdot k2}, \sqrt{1-(1-\sin \lambda \eta_1)^{k1 \cdot k2}} \right\}$

$= (\sin_\lambda d_1)^{k_1 k_2}$

Proof of theorem 3.1.3:

$\sin_{\lambda_1} d = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \left\{ \sqrt{1-\sin \lambda_1 \sqrt{(1-\gamma^2)}}, \sqrt{\sin \lambda_1 \eta} \right\}$

$\sin_{\lambda_2} d = \bigcup_{\gamma_2 \in h_2, \eta_2 \in g_2} \left\{ \sqrt{1-\sin \lambda_2 \sqrt{(1-\gamma^2)}}, \sqrt{\sin \lambda_2 \eta} \right\}$

Since we have $\lambda_1 \leq \lambda_2 \Rightarrow \sin \lambda_1 \sqrt{(1-\gamma^2)} \leq \sin \lambda_2 \sqrt{(1-\gamma^2)}$

hence, $\sqrt{1-\sin \lambda_1 \sqrt{(1-\gamma^2)}} \geq \sqrt{1-\sin \lambda_2 \sqrt{(1-\gamma^2)}}$

and $\sin \lambda_1 \eta \leq \sin \lambda_2 \eta \Rightarrow \sqrt{\sin \lambda_1 \eta} \leq \sqrt{\sin \lambda_2 \eta}$

hence, $\sin_{\lambda_1} d \geq \sin_{\lambda_2} d$

Proof of theorem 3.1.4:

Since $\gamma_1 \leq \gamma_2 \Rightarrow \gamma_1{}^2 \leq \gamma_2{}^2 \Rightarrow \sqrt{(1-\gamma_1{}^2)} \geq \sqrt{(1-\gamma_2{}^2)}$

$\sin \lambda \sqrt{(1-\gamma_1{}^2)} \geq \sin \lambda \sqrt{(1-\gamma_2{}^2)} \Rightarrow \sqrt{1-\sin \lambda \sqrt{(1-\gamma_1{}^2)}} \leq \sqrt{1-\sin \lambda \sqrt{(1-\gamma_2{}^2)}}$

Now $\eta_1 \geq \eta_2$ so $\sin \lambda \eta_1 \geq \sin \lambda \eta_2 \Rightarrow \sqrt{\sin \lambda \eta_1} \geq \sqrt{\sin \lambda \eta_2}$

Hence $\sin_\lambda d_1 \leq \sin_\lambda d_2$

Appendix C. Proof of property 3.2.1

Since all $d_i = (h_i, g_i) = d$ then by theorem 3.2.1

THPFWA $(d_1, d_2 \ldots\ldots d_n) =$

$$\bigcup_{\gamma_1\epsilon h_1,\gamma_2\epsilon h_2 \ldots\ldots \gamma_n\epsilon h_n, n_1\epsilon g_1, n_2\epsilon g_2 \ldots\ldots n_n\epsilon g_n} \left\{\left\{\sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}}\right\},\left\{\prod_{i=1}^{n}\sqrt{(\sin\lambda\eta_i)}^{\omega_i}\right\}\right\}$$

$$= \bigcup_{\gamma_1\epsilon h_1, n_1\epsilon g_1} \left\{\left\{\sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda\sqrt{1-\gamma^2}\right]^{\omega_i}}\right\},\left\{\prod_{i=1}^{n}\sqrt{(\sin\lambda\eta)}^{\omega_i}\right\}\right\}$$

$$= \bigcup_{\gamma_1\epsilon h_1, n_1\epsilon g_1} \left\{\left\{\sqrt{1-\left[\sin\lambda\sqrt{1-\gamma^2}\right]^{\sum_{i=1}^{n}\omega_i}}\right\},\left\{\sqrt{(\sin\lambda\eta)}^{\sum_{i=1}^{n}\omega_i}\right\}\right\}$$

$$= \bigcup_{\gamma_1\epsilon h_1, n_1\epsilon g_1} \left\{\left\{\sqrt{1-\left[\sin\lambda\sqrt{1-\gamma^2}\right]}\right\},\left\{\sqrt{(\sin\lambda\eta)}\right\}\right\}$$

$= \sin_{\lambda} d$

Proof of property 3.2.2:

Let $\gamma^- = \sqrt{1-\left[\sin\lambda\sqrt{1-\gamma^{-2}}\right]}$, $\gamma^+ = \sqrt{1-\left[\sin\lambda\sqrt{1-\gamma^{+2}}\right]}$, $\eta^- = \sqrt{(\sin\lambda\eta^-)}$, $\eta^+ = \sqrt{(\sin\lambda\eta^+)}$, $\gamma = \sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}}$ and $\eta = \prod_{i=1}^{n}\sqrt{(\sin\lambda\eta_i)}^{\omega_i}$. Then $d = (\gamma, \eta)$, $d^+ = (\gamma^+, \eta^+)$ and $d^- = (\gamma^-, \eta^-)$.

For all j we have $\min_j h_j \le \gamma_j \le \max_j h_j$, $\min_j g_j \le \eta_j \le \max_j g_j$ and $\sum_{i=1}^{n}\omega_i = 1$.

Assume THPFWA ($d_1, d_2 \ldots\ldots d_n$) = $\sin_{\lambda} d$, THPFWA (d^+) = $\sin_{\lambda} d^+$ and THPFWA (d^-) = $\sin_{\lambda} d^-$

$$\sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda\sqrt{1-\gamma^{-2}}\right]^{\omega_i}} \le \sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}} \le \sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda\sqrt{1-\gamma^{+2}}\right]^{\omega_i}}$$

$$\sqrt{1-\left[\sin\lambda\sqrt{1-\gamma^{-2}}\right]^{\sum_{i=1}^{n}\omega_i}} \le \sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}} \le \sqrt{1-\left[\sin\lambda\sqrt{1-\gamma^{+2}}\right]^{\sum_{i=1}^{n}\omega_i}}$$

$$\sqrt{1-\left[\sin\lambda\sqrt{1-\gamma^{-2}}\right]} \le \sqrt{1-\prod_{i=1}^{n}\left[\sin\lambda\sqrt{1-\gamma_i{}^2}\right]^{\omega_i}} \le \sqrt{1-\left[\sin\lambda\sqrt{1-\gamma^{+2}}\right]}$$

$i.e.,\ \gamma^- \le \gamma \le \gamma^+$

Similarly, $\prod_{i=1}^{n}\sqrt{(\sin\lambda\eta^+)}^{\omega_i} \le \prod_{i=1}^{n}\sqrt{(\sin\lambda\eta_i)}^{\omega_i} \le \prod_{i=1}^{n}\sqrt{(\sin\lambda\eta^-)}^{\omega_i}$

$$\sqrt{(\sin\lambda\eta^+)}^{\sum_{i=1}^{n}\omega_i} \le \prod_{i=1}^{n}\sqrt{(\sin\lambda\eta_i)}^{\omega_i} \le \sqrt{(\sin\lambda\eta^-)}^{\sum_{i=1}^{n}\omega_i}$$

$$\sqrt{(\sin\lambda\eta^+)} \le \prod_{i=1}^{n}\sqrt{(\sin\lambda\eta_i)}^{\omega_i} \le \sqrt{(\sin\lambda\eta^-)}$$

$i.e.,\quad \eta^+ \le \eta_j \le \eta^-$

According to score function S (THPFWA ($d_1, d_2 \ldots\ldots d_n$)) = $\sum_{\gamma\epsilon h}\gamma^2 / *h$ - $\sum_{\eta\in g}\eta^2 / *g$

S (d^+) = $\sum_{\gamma^+\epsilon h^+}\gamma^{+2} / *h^+$- $\sum_{\eta^+\in g^+}\eta^{+2} / *g^+ = \gamma^{+2}$ - η^{+2}

S (d^-) = $\sum_{\gamma^-\epsilon h^-}\gamma^{-2} / *h^-$- $\sum_{\eta^-\in g^-}\eta^{-2} / *g^- = \gamma^{-2}$ - η^{-2}

S (THPFWA ($d_1, d_2 \ldots\ldots d_n$)) = $\sum_{\gamma\epsilon h}\gamma^2 / *h$ - $\sum_{\eta\in g}\eta^2 / *g \le \sum_{\gamma\epsilon h}\gamma^{+2} / *h$ - $\sum_{\eta\in g}\eta^{+2} / *g$

$\le \gamma^{+2}$ - η^{+2} = S (d^+)

S (THPFWA ($d_1, d_2 \ldots\ldots d_n$)) = $\sum_{\gamma \epsilon h} \gamma^2 / * h$ - $\sum_{\eta \in g} \eta^2 / * g \geq \sum_{\gamma \epsilon h} \gamma^{-2} / * h$ - $\sum_{\eta \in g} \eta^{-2} / * g$

$\geq \gamma^{-2}$ - η^{-2} = S (d^-)

Hence, S (d^-) ≤ S (THPFWA ($d_1, d_2 \ldots\ldots d_n$)) ≤ S ($d^+$) ⇒ $\sin_\lambda d^- \leq$ THPFWA ($d_1, d_2 \ldots\ldots d_n$) $\leq \sin_\lambda d^+$

Proof of property 3.2.3: Same as above.