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Determining the price elasticity of demand with and without memory effects using fractional order derivatives: A numerical simulation approach

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A B S T R A C T

Demand elasticity is the sensitivity of changes in the number of goods demanded by consumers due to changes in the price of goods. This paper compares the price elasticity of demand with and without memory effect using fractional-order derivatives. This study is designed using the development theory of fractional derivatives for the economic field in determining the price elasticity of demand. The result of numerical simulation using the value of α and p indicated that the price elasticity of demand with memory effect is more accurate than without the memory effect. Furthermore, this study concluded that the price elasticity of demand does not only depend on the latest price (current price) but changes in all prices from a specific time interval. The findings of this study suggest future studies can examine the phenomenon of market equilibrium using fractional-order derivatives.

1. Introduction

Fractional orders have received more attention due to their potential applications in economics and management, such as Chinese economic growth models (Ming, Wang, & Fečkan, 2019), Spanish economic growth models (Inés Tejado, Valério, & Valério, 2015), Portuguese economic growth models (Inés Tejado, Valério, & Valério, 2014), European union economic growth models (Tejado, Perez, & Valerio, 2018), economic production quantity model (Rahaman et al., 2020), a financial system with market confidence (Xin & Zhang, 2015), western global economic downturn (Machado & Mata, 2015), IS-LM Macroeconomic System (Ma & Ren, 2016), regional economic income multiplication capability (Fang, 2020), regional economic system impact factors (Zhang, Fu, & Morris, 2019), business cycle model (Xie, Wang, & Meng, 2019), investment incentive (Xin & Li, 2013) commercial and rural banks in Indonesia (Khan, Azizah, & Ullah, 2019).

Some literature related to price elasticity can be seen in (Andruszkiewicz, Lorenc, & Weychan, 2019, 2020; Babbel, 1985; Kim & Kwak, 2004; Petricek, Chalupa, & Chadt, 2020; Rashid & Mitra, 1999; Sukawattanapornkul, Siritaratiwat, Khunkitti, Surawanitkun, & Chatthaworn, 2019; Tarasova & Tarasov, 2016). Andruszkiewicz, Lorenc, & Weychan (2019) proposed a new method of determining the price elasticity of electricity demand based on a comparison of load profiles of customers and the price elasticity of demand of households using zonal tariffs and its impact on the hourly load of the power system (Andruszkiewicz et al., 2020). Sukawattanapornkul et al. (2019) presented the price elasticity of the demand model to evaluate the change in demand after changing electricity rates. Petricek et al. (2020) identification of consumer behavior based on price elasticity. Babbel (1985) analyzed the price elasticity of demand for whole life insurance. Rashid & Mitra (1999) investigated price elasticity in optimal cash discount rate in credit policy. Kim & Kwak (2004) described the price elasticity in the parcel service market. Tarasova & Tarasov (2016) analyzed elasticity for economical processes with

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memory using fractional order. However, the presentation of literature related to price elasticity using fractional order is still very minimal and needs further development.

The dynamic market equilibrium price function is determined using a second-order linear differential equation. However, derivative orders commonly used (including those given to students in formal lectures) are natural numbers. However, it is a big challenge for mathematicians to expand the order of these derivatives into rational numbers and even real numbers. Literature that comprehensively discusses the properties and generalizations of derivatives and fractional integrals can refer to (Chen, Rashid, Noor, Ashraf, & Chu, 2020; Sambas, He, et al., 2020; Sambas, Vaidyanathan, et al., 2020). This paper examines the method of determining the price elasticity of demand using the fractional derivative, which describes the level of memory reduction over a specific time interval. In this case, we use $\alpha = \frac{1}{2}$ and $\alpha = 1$ to observe. This study's main contribution is to determine the price elasticity of demand with and without memory effect. The rest of this paper is organized as follows. Section 2 describes the background theory of fractional calculus. The methodology of the research is shown in Section 3. Results and discussion are discussed in Section 4. Finally, the conclusions of this paper are summarized in Section 5.

2. Background Theory

2.1. Fractional Calculus

Fractional calculus is a field of mathematics that studies the theory of integral and derivative fractals (any), which is an extension of derivatives and integers of integer order (Diethelm & Ford, 2002). Fractional calculus was first introduced by L 'Hospital and Leibniz in 1695, which discussed the n$^{\text{th}}$ derivative of a simple polynomial function $f(x) = x^m$ that is $\frac{d^n x^m}{dx^n}$ for $n = \frac{1}{2}$. Fractional operators and their applications have essential roles in various fields of science such as mechanics, electro, chemistry, biology, economics, and control theory (Sambas et al., 2021; Vaidyanathan, Feki, Sambas, & Lien, 2018; Vaidyanathan, Sambas, Kacar, & Cavusoglu, 2019). The fractional derivative of a function $y = f(x)$ can be defined as $\frac{d^n f(x)}{dx^n}$, where n is any real order. For example, $y = x^m$ where m is positive integers, Lacroix denotes derivatives to-n from the function $y = f(x)$ that is:

$$\frac{d^n y}{dx^n} = \frac{d^n f(x)}{dx^n} = \frac{d^n x^m}{dx^n} = \frac{m!}{(m-n)!} x^{m-n} \quad , m \geq n \tag{1}$$

By using the Gamma symbol (Γ) to replace the definition of factorial functions, the Lacroix formula in Eq. (1) is written as follow:

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)!}{\Gamma(m-n+1)!} x^{m-n} \tag{2}$$

The fractional derivative defined by Lacroix is still being developed by several other scientists such as Euler, Laplace, Fourier, Abel, Riemann, Liouville, and Caputo. For example, $y = f(x)$ is a function that is defined for x> 0. The integral form from 0 to x can be written as:

$$Jf(x) = \int_0^x f(t)dt \tag{3}$$

Eq. (3) is reintegrated from 0 to t, and it is obtained:

$$(J^2 f)(x) = \int_0^x \left(\int_0^t f(s)ds \right) dt$$

This integration can be repeated until the nth integral is obtained:

$$(J^n f)(x) = \int_0^x \int_0^{tn} \dots \int_0^{t2} f(t_1)dt_1 dt_2 \dots dt_n \tag{4}$$

By using the Cauchy formula for repeated integrals, Eq. (4) can be written as:

$$(J^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)dt \tag{5}$$

This Eq. (5) then becomes the basis for constructing the definition of fractional order integrals (any real). The definition of the Gamma function is:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt, \qquad z \,\epsilon R^+ \tag{6}$$

Gamma function has basic properties. For example, in Eq. (6), the Gamma function for a positive integer z is written as:

$$\Gamma(z) = (z-1)!,\ z\,\epsilon R^+ \tag{7}$$

Based on Eq. (7), the Gamma function is an extension of the factorial function. Although the Gamma function is defined for *z* > 0, it is also possible to develop Gamma function definitions for all *z* negative real numbers, i.e.,:

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1)$$

Eq. (5) is only limited to n integers due to factorial functions. The Gamma function is an extension of the factorial function for all real numbers so that it can substitute for factorial functions as in Eq. (7). Therefore, Eq. (5) is written as:

$$(J^\alpha f)(x) = \frac{1}{\Gamma\,(\alpha)}\int_0^x (x-t)^{\alpha-1}\, f(t)dt, \tag{8}$$

Fractional integral definitions can also be written in fractional differential operators with α non-negative real numbers.

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, f(t)$$

Riemann-Liouville formulates the definition of the fractional integral in equation (8), so it is often known as the Riemann-Liouville fractional integral. In general, the fractional integral operator is then used to define the fractional derivative α of the function *f (x)*, namely $D^\alpha f(x)$.

2.2 Integral and Fractional Derivatives

According to Baleanu and Agarwal (2021), fractional integrals and derivatives are integrals and derivatives of fractional order. Several approaches to denoting fractional-order derivatives include Riemann-Liouville, Caputo, and Grunwald-Letnikov. The Riemann-Liouville fractional derivative is the most commonly used approach of these approaches. Based on equation (8), the Riemann-Liouville fractional integral can be defined.

Definition 2.1. Let α be a real number, the fractional integral of the order α of th:

$$J^\alpha f(x) = D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, f(t)\, dt, \tag{9}$$

With $\alpha > 0$. As for $\alpha \geq 0$, and $\beta \geq 0$, the fractional integrals proposed by Riemann-Liouville have the following characteristics:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

Definition 2.2. Riemann-Liouville fractional derivatives are defined as:

$$D_{x_0}^\alpha f(x) = \frac{d^n}{dx^n}[J^{n-\alpha} f(x)] \tag{10}$$

$$D_{x_0}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\left[\int_{x_0}^x (x-t)^{n-\alpha-1}\, f(t)\, dt\right] \tag{11}$$

with α any order, $n - 1 \leq \alpha < n$, $n \in Z^+$, x_0 lower limit, $x_0 < x$, $x > 0$, and $D_{x_0}^\alpha$. The fractional derivative operator of order α with the lower limit x_0. Riemann-Liouville fractional derivatives have the drawback of real-world modeling events because they require the definition of boundary conditions and initial values of fractional order. The boundary conditions and initial fractional-order values do not have a clear interpretation in real life, so Caputo fractional derivatives are introduced as alternatives.

The positive order fractional derivative defined by Caputo is a modification of Riemann-Liouville.

Definition 2.3. Let α be a real number, and n-1 < α ≤ n, where n is a natural number, the fractional-order of the order α of *f (x)* for *x*:

$$D^{\alpha} f(x) = J^{n-\alpha} D^{n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{x} (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt \tag{12}$$

Definition 2.3 means that the fractional derivative operator is ordered α with the lower limit $x_0 = 0$ of the function *f (x)* with respect to *x* is written as $D^{\alpha} f(x)$. Thus the fractional integral of order α is an anti-fractional sequence of order (- α) with the lower limit $x_0 = 0$ from function $f(x)$ to x which is written as $D^{-\alpha} f(x)$, so it applies $J^{\alpha} f(x) = D^{-\alpha} f(x)$. Furthermore, to calculate the fractional derivative from the sum of three or more functions, we can use the Caputo Fractional derivative properties.

$$\begin{aligned} D^{\alpha}[k_1\, f(t) \;+\; k_2\, g(t) \;+\; k_3\, h(t)\,] &= D^{\alpha} k_1\, f(t) \;+\; D^{\alpha}\, k_2\, g(t) \;+\; D^{\alpha}\, k_3\, h(t) \\ &= k_1\, D^{\alpha} f(t) \;+\; k_2\, D^{\alpha}\, g(t) \;+ k_3 D^{\alpha}\, h(t) \end{aligned} \tag{13}$$

2.3 Elasticity

The elasticity is a tool used to measure the level of sensitivity of consumers and producers to price changes (Baleanu & Agarwal, 2021). So, in broad terms, understanding elasticity is the degree of change in demand or supply to price changes. Elasticity is divided into two, namely, the elasticity of demand and the elasticity of supply.

2.3.1 Demand Elasticity

In measuring how much consumers react to changes in prices and other factors, economists use the concept of elasticity. Demand elasticity is the sensitivity of changes in the number of goods requested by consumers to changes in the price of goods. This elasticity can tell producers what happened to their sales revenue, whether to increase or decrease the number of goods sold after applying a price change strategy. In general, the elasticity of demand can be divided into 3 (Škovránek, Podlubny, & Petráš, 2012), namely: price elasticity of demand, income elasticity of demand, and cross price elasticity of demand. In the elasticity of demand besides price, several other variables influence demand, among others, e.g., (i) price of other products will have a positive effect if the price rises for substitute products and negative if complementary products; (ii) consumer income can have a positive effect, but for inferior products, it can have a negative impact; (iii) product price expectations in the future have a positive effect; (iv) expectations of future consumer income have a positive effect; (iv) expectations of product availability in the future have a negative effect; (v) consumer tastes have a positive effect; (vi) the number of potential consumers has a positive effect; (vii) expenditures have a positive effect and product attributes also have a positive effect on demand.

2.3.2 Price Elasticity of Demand

The elasticity of demand is presented in the form of an elasticity coefficient which is defined as a pointer to illustrate how much change in the number of goods demanded compared to price changes (Andreyeva, Long, & Brownell, 2010). The elasticity of demand can be explained as follows:

a. Elastic Uniter (E = 1); If the price goes up / down by 1%, then the demand will go up / down by 1% (the percentage of the number of goods demanded is the same as the percentage change in price).
b. Elastic (E > 1) ; If the price goes up / down by 1%, then the demand will go up / down by more than 1% (the percentage change in the number of goods demanded is higher than the percentage change in price; demand is susceptible to price changes)
c. Inelastic (E < 1); If the price goes up / down by 1%, then the demand will go down / up by less than 1% (the percentage change in the amount requested is smaller than the percentage change in price; demand is not sensitive to price changes).
d. Perfect inelasticity (E = 0); If demand is not responsive to price changes, so whatever the market price is, then the number of goods demanded remains (demand curve parallel to the vertical axis/price axis)
e. Perfect Elasticity (E = ~); If the consumer wants to buy any amount of goods offered at a certain price level, then the demand curve is parallel to the horizontal axis / the number of requests.

3. Methodology

The research method adopted is to develop a theory from previous research to be applied more broadly. This research was conducted in two stages. First, this study reviews some methods of solving fractional derivatives that have been achieved in previous studies. Second, this study applies them to dynamic economics cases to compare the results and make conclusions, where the methods studied to produce the same solution.

4. Results and Discussion

4.1 Fractional Derivative

According to Baleanu and Agarwal (2021), the α-ordered fractional integral of a simple polynomial function in the form of *f (x) = x^m* can be expressed in the form of multiplication of the Gamma function with the polynomial function as follows.

From equation (9), for the lower limit $x_0 = 0$ and $f(x) = x^m$ was obtained:

$$J^\alpha f(x) = D^{-\alpha} x^m = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^m \, dt \tag{14}$$

So, it can be calculated $D^{-\alpha}x^m$, where $\alpha > 0, m > -1$, as follows:

$$D^{-\alpha}x^m = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t^m dt = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\frac{t}{x})^{\alpha-1} x^{\alpha-1} t^m dt = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1} x^{\alpha-1} (xu)^m x \, du,$$

with $u = \frac{t}{x}$

$$D^{-\alpha}x^m = \frac{1}{\Gamma(\alpha)} x^{m+\alpha} \int_0^1 (1-u)^{\alpha-1} (u)^m du = \frac{1}{\Gamma(\alpha)} x^{m+\alpha} B(m+1,\alpha) = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} x^{m+\alpha}.$$

So, the fractional integral of the α order of the simple shaped polynomial function $f(x) = x^m$ can be stated in the following theorem.

Theorem 4.1. The fractional integral of the α order of the shaped polynomial function $f(x) = x^m$ is:

$$D^{-\alpha}x^m = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} x^{m+\alpha}, for \ \alpha > 0, m > -1, x > 0 \tag{15}$$

Eq. (14) informs us that the fractional integral of a constant *k* of order α is:

$$D^{-\alpha}k = \frac{k}{\Gamma(\alpha+1)} x^\alpha \tag{16}$$

Furthermore, it is assumed that $\alpha = n - u$. where $0 < \alpha < 1$ and n is the smallest integer greater than u. Then, the child function $f(x)$ orde *u* is:

$$D^u f(x) = D^n[D^{-\alpha} f(x)] \tag{17}$$

For example, find the fractional derivative function α of *f (x) = x^m*, where *m ≥ 0* using **Theorem 4.1**, then suppose *u = n-α*, where *0 <u <1*. Then we have *n = 1* and *u = 1-α*. Then,

$$D^\alpha f(x) = D^1[D^{-(1-\alpha)} f(x)] = D^1[D^{-(1-\alpha)} x^m] = D^1[\frac{\Gamma(m+1)}{\Gamma((m-\alpha+1)+1)} x^{m-\alpha+1}]$$
$$= [(m-\alpha+1)\frac{\Gamma(m+1)}{(m-\alpha+1)\Gamma(m-\alpha+1)} x^{m-\alpha}] = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha},$$

So, the fractional derivative *α* of a simple polynomial function *f (x) = x^m* can be expressed in the following theorem.

Theorem 4.2. The fractional derivative *α* of the polynomial function in the form of *f (x) = x^m* is:

$$D^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}. untuk \ m \geq 0. 0 < \alpha < 1 \tag{18}$$

4.2 Determining the Price Elasticity of Demand

Definition of standard price point elasticity of demand when $t = t_0$, which is shown by the Eq. (19):

$$E(Q(t); p(t); t_0) = \left(\frac{p(t)}{Q(t)} \frac{dQ/dt}{dp/dt}\right)_{t=t_0} = \left(\frac{p}{Q} \frac{dQ}{dp}\right)_{t=t_0} \tag{19}$$

where Q is the number of requests and p is the price of an item. In Eq. (19), it is assumed that elasticity depends only on the current price when $t = t_0$, i.e., the lowest price around t_0. In general, Eq. (19) can only be used when all buyers have total amnesia, i.e., The interpretation of total amnesia is not too concerned about changes in previous prices. In this section, a method for determining the elasticity of demand prices is proposed using the fractional order α, with $0 < \alpha < 1$, which involves an element of consumer memory. Fractional elasticity depends on a specific time interval and or price range, in addition to the consumer's memory parameters. For example, we assume a parameter α, which is indicated by the level of consumer memory (memory loss) during a specified time interval. Thus, if $\alpha = 1$, the order of memory loss is 1.

4.2.1 Calculating the Price Elasticity of Demand for $\alpha = 1$

Calculate the elasticity of the demand price for $\alpha = 1$ of the demand functions for a product below:

$$Q(p) = k_0 + k_1 p + k p^2, \tag{20}$$

where p is the unit price, and $Q(p)$ is the amount of demand at the time of price p.

Eq. (19) can be written as follows,

$$E(p) = \left(\frac{p}{Q(p)}\right)\left(\frac{dQ(p)}{dp}\right) \tag{21}$$

Thus, the price elasticity of demand price for $\alpha = 1$ is:

$$E(p) = \frac{p}{Q(p)}(k_1 + 2k_2 \mathrm{p}) = \frac{k_1 p + 2 k_2 p^2}{k_0 + k_1 p + k_2 p^2}. \tag{22}$$

4.2.2 Calculating Price Elasticity of Demand for $\alpha = \frac{1}{2}$

We will discuss examples of calculating the fractional price elasticity of demand using the Riemann-Liouville Fractional Derivative Method, Caputo Fractional Derivative Method, and the Fractional Derivative Theorem Method.

Example 4.1

The following item request functions are known:

$$Q(p) = k_0 + k_1 p + k_2 p^2 \tag{23}$$

where p is the unit price $Q(p)$ is the amount of demand when the price is at time p. Determine fractional elasticity by order $\alpha = \frac{1}{2}$. To solve Example 4.1, three methods are used, namely the Riemann-Liouville fractional derivative method, the Caputo fractional derivative method, and **Theorem 4.2**.

4.3 Riemann-Liouville Fractional Derivative

Riemann-Liouville fractional derivative defined as:

$$D_{x_0}^{\alpha} f(x) = \frac{d^n}{dx^n}[J^{n-\alpha} f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n}\left[\int_{x_0}^{x} (x-t)^{n-\alpha-1} f(t)\, dt\right], \tag{24}$$

with any order $\alpha, n - 1 \leq \alpha < n. n \in Z^+$, with the lower limit x_0, $x_0 < x. x > 0$, and D^{α} fractional differential operator of order α with a lower bound $x_0 = 0$. So, the fractional sequence is ordered $\alpha = ½$ from $Q(p) = k_0 + k_1 p + k_2 p^2$ is:

$$D^{\alpha} Q(p) = D^{\alpha} k_0 + D^{\alpha} k_1 p + D^{\alpha} k_2 p^2$$

$$D^{\frac{1}{2}}Q(p) = k_0\, D^{\alpha}\, p^0 + k_1\, D^{\frac{1}{2}}\, p^1 + k_2\, D^{\frac{1}{2}}\, p^2$$

with Caputo's fractional derivative of the constant k_0 i.e., $D^{\alpha}\, k_0 = 0$, is obtained:

$$D^{\frac{1}{2}}Q(p) = 0 + \frac{d}{dp}\left[\frac{k_1}{\Gamma\left(1-\frac{1}{2}\right)}\int_0^p (p-t)^{1-½-1}\, t\, dt\right] + \frac{d}{dp}\left[\frac{k_2}{\Gamma\left(1-\frac{1}{2}\right)}\int_0^p (p-t)^{1-\frac{1}{2}-1}.\, t^2\, dt\right]$$

$$D^{\frac{1}{2}}Q(p) = \frac{d}{dp}\left[\frac{k_1}{\Gamma\left(\frac{1}{2}\right)}\int_0^p \frac{t}{(p-t)^{\frac{1}{2}}}\, dt\right] + \frac{d}{dp}\left[\frac{k_2}{\Gamma\left(\frac{1}{2}\right)}\int_0^p \frac{t^2}{(p-t)^{\frac{1}{2}}}\, dt\right]$$

For example, $t = pu$, for $t = 0 \rightarrow u = 0$, $dt = p\, du$, for $t = p \rightarrow u = 1$, then

$$D^{\frac{1}{2}}Q(p) = \frac{d}{dp}\left[\frac{k_1}{\sqrt{\pi}}\int_0^1 \frac{pu}{(p-pu)^{\frac{1}{2}}}\, .pdu\right] + \frac{d}{dp}\left[\frac{k_2}{\sqrt{\pi}}\int_0^1 (p-pu)^{-0,5}.\, u^2.\, p^2\, .\, p\, du\right]$$

$$D^{\frac{1}{2}}Q(p) = \frac{d}{dp}\left[\frac{k_1}{\sqrt{\pi}}\int_0^1 \frac{p^2 u}{p^{½}\, (1-u)^{\frac{1}{2}}}\, du\right] + \frac{d}{dx}\left[\frac{k_2}{\sqrt{\pi}}\int_0^1 p^{-\,0,5}.\ (1-u)^{-0,5}\, .\, u^2.\, p^3\, du\right]$$

$$D^{\frac{1}{2}}Q(p) = \frac{d}{dp}\left[\frac{k_1 p^{\frac{3}{2}}}{\sqrt{\pi}}\int_0^1 u^1\, (1-u)^{-½}\, du\right] + \frac{d}{dp}\, k_2\left[\frac{1}{\sqrt{\pi}}\, p^{2,5}\int_0^1\ (1-u)^{-0,5}\, .\, u^2\, du\right]$$

$$D^{\frac{1}{2}}Q(p) = \frac{d}{dp}\left[\frac{k_1 p^{\frac{3}{2}}}{\sqrt{\pi}}\int_0^1 u^{2-1}\, (1-u)^{½-1}\, du\right] + \frac{d}{dp}\left[\frac{k_2}{\sqrt{\pi}}\, p^{\frac{5}{2}}\int_0^1\ (1-u)^{0,5-1}\, .\, u^{3-1}\, du\right]$$

$$D^{\frac{1}{2}}Q(p) = \frac{d}{dp}\left[\frac{k_1 p^{\frac{3}{2}}}{\sqrt{\pi}}\, B\left(2, \frac{1}{2}\right)\right] + \frac{d}{dp}\left[\frac{k_2}{\sqrt{\pi}}\, p^{\frac{5}{2}}\, .\, B\left(\frac{1}{2}\, .\, \frac{6}{2}\right)\right]$$

$$D^{\frac{1}{2}}Q(p) = \frac{d}{dp}\left[\frac{p^{\frac{3}{2}}}{\sqrt{\pi}}\ \frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}\right] + \frac{d}{dx}\left[\frac{k_2}{\sqrt{\pi}}\, p^{\frac{5}{2}}\, .\, \frac{\Gamma\left(\frac{1}{2}\right).\Gamma(3)}{\Gamma\left(\frac{7}{2}\right)}\right]$$

$$D^{\frac{1}{2}}Q(p) = \frac{d}{dp}\left[\frac{k_1 p^{\frac{3}{2}}}{\sqrt{\pi}}\ \frac{\sqrt{\pi}}{\frac{3\sqrt{\pi}}{4}}\right] + \frac{d}{dp}\left[\frac{k_2}{\sqrt{\pi}}\, p^{\frac{5}{2}}\ \frac{\sqrt{\pi}\ (2)}{\frac{15}{8}\sqrt{\pi}}\right]$$

$$D^{\frac{1}{2}}Q(p) = \frac{4k_1}{3\sqrt{\pi}}\,\frac{d}{dp}\left[p^{\frac{3}{2}}\right] + \frac{16\, k_2}{15\sqrt{\pi}}\,\frac{d\,(p^{\frac{5}{2}})}{dp}$$

$$D^{\frac{1}{2}}Q(p) = \frac{4k_1}{3\sqrt{\pi}}\,\frac{3}{2}\, p^{½} + \frac{16\, k_2}{15\sqrt{\pi}}\, .\, \frac{5}{2}\, p^{\frac{3}{2}}$$

$$D^{\frac{1}{2}}Q(p) = \frac{2\, k_1 p^{½}}{\sqrt{\pi}} + \frac{8\, k_2\sqrt{p^3}}{3\,\sqrt{\pi}}$$

$$D^{\frac{1}{2}}Q(p) = 2\, k_1\sqrt{\frac{p}{\pi}} + \frac{8}{3}\, k_2\sqrt{\frac{p^3}{\pi}}$$

Furthermore, fractional elasticity with order is sought $\alpha = \frac{1}{2}$, was obtained

$$E_{\alpha}\,(Q(p), p) = \frac{p}{Q(p)}\,\frac{d^{\alpha}Q(p)}{d^{\alpha}p}$$

$$E_{\frac{1}{2}}\,(Q(p), p) = \frac{p}{Q(p)}\,\frac{d^{\frac{1}{2}}Q(p)}{d^{\frac{1}{2}}p}$$

$$E_{\frac{1}{2}}\,(Q(p), p) = \frac{p}{k_0 + k_1 p + k_2 p^2}\, .\, 2\, k_1\sqrt{\frac{p}{\pi}} + \frac{8}{3}\, k_2\sqrt{\frac{p^3}{\pi}} = \frac{2pk_1\sqrt{\frac{p}{\pi}} + \frac{8}{3}\, pk_2\sqrt{\frac{p^3}{\pi}}}{k_0 + k_1 p + k_2 p^2} \tag{25}$$

4.4 Caputo Fractional Derivative

Caputo's fractional derivatives are defined as in Eq. (12), then obtained:

$$D^{\alpha}Q(p) = D^{\alpha}\, k_0 + \; D^{\alpha}k_1 p + D^{\alpha}\, k_2 p^2$$

$$D^{\frac{1}{2}}Q(p) = \; k_0\, D^{\alpha}\, p^0 + k_1\, D^{\frac{1}{2}}\, p + k_2\, D^{\frac{1}{2}}\, p^2,$$

$$D^{\frac{1}{2}}Q(p) = \frac{k_0}{\Gamma\,(1-\frac{1}{2})} \int_0^p (p-t)^{1-\frac{1}{2}-1}\, f'(t)dt + \frac{k_1}{\Gamma\,(1-\frac{1}{2})} \int_0^p (p-t)^{1-\frac{1}{2}-1}\, f'\,(t)\,dt + \frac{k_2}{\Gamma\,(1-\frac{1}{2})} \int_0^p (p-t)^{1-\frac{1}{2}-1}\, f'\,(t)\,dt$$

$$D^{\frac{1}{2}}Q(p) = \frac{1}{\Gamma\,(\frac{1}{2})} \int_0^p (p-t)^{-½} \cdot 0 \cdot dt \; + \frac{k_1}{\Gamma\,(\frac{1}{2})} \int_0^p \frac{1}{\sqrt{p-t}} \cdot 1 \cdot dt \; + \frac{k_2}{\Gamma\,(\frac{1}{2})} \int_0^p \frac{1}{\sqrt{p-t}}\, .2t.dt$$

$$D^{\frac{1}{2}}Q(p) = 2k_1\sqrt{\frac{p}{\pi}} + \frac{8}{3}\, k_2\sqrt{\frac{p^3}{\pi}}$$

Next, we look for fractional elasticity with order α = $\frac{1}{2}$, was obtained:

$$E_{\frac{1}{2}}\,(Q(p),p) = \frac{p}{Q(p)}\, \frac{d^{\frac{1}{2}}Q(p)}{d^{\frac{1}{2}}p} = \frac{2pk_1\sqrt{\frac{p}{\pi}} + \frac{8}{3}pk_2\sqrt{\frac{p^3}{\pi}}}{k_0 + k_1 p + k_2 p^2} \tag{26}$$

Solution using **Theorem 4.2. is** obtained:

$$Q(p) \; = k_0 + k_1 p + \; k_2 p^2$$

$$D^{\alpha}Q(p) = D^{\alpha}k_0 + \; D^{\alpha}k_1 p + D^{\alpha}k_2 p^2$$

$$D^{\frac{1}{2}}Q(p) = \; D^{\frac{1}{2}}\, k_0 + k_1\, D^{\frac{1}{2}}\, p + k_2\, D^{\frac{1}{2}}\, p^2,$$

$$D^{\frac{1}{2}}Q(p) = 0 + k_1\, \frac{\Gamma\,(1+1)}{\Gamma\,(1-\frac{1}{2}+1\,)}\, p^{1-\frac{1}{2}} + k_2\, \frac{\Gamma\,(2+1)}{\Gamma\,(2-\frac{1}{2}+1\,)}\, p^{2-\frac{1}{2}}$$

$$D^{\frac{1}{2}}Q(p) = \; k_1\, \frac{\Gamma\,(2)}{\Gamma\,(\frac{3}{2})}\, p^{\frac{1}{2}} + k_2\, \frac{\Gamma\,(3)}{\Gamma\,(\frac{5}{2})}\, p^{\frac{3}{2}}$$

$$D^{\frac{1}{2}}Q(p) = 2\; k_1\, \sqrt{\frac{p}{\pi}} \; + \; \frac{8}{3}k_2\sqrt{\frac{p^3}{\pi}}$$

Next, we look for fractional elasticity with order $\alpha = \frac{1}{2}$, was obtained:

$$E_{\frac{1}{2}}\,(Q(p),p) = \frac{p}{Q(p)}\, \frac{d^{\frac{1}{2}}Q(p)}{d^{\frac{1}{2}}p} = \frac{2pk_1\sqrt{\frac{p}{\pi}} + \frac{8}{3}pk_2\sqrt{\frac{p^3}{\pi}}}{k_0 + k_1 p + k_2 p^2} \tag{27}$$

Based on Eq. (25) to Eq. (26), the determination of price elasticity of demand using either the Riemann-Liouville Fractional Derivative Method, Caputo's Fractional Derivative Method, and Theorem 4.2. give the same results. This indicates that the three methods can be used to calculate the fractional of price elasticity of demand.

Example 4.2

The following item request functions are known:

$$Q(p) = 180 - 9p^2. \tag{28}$$

Determine the price elasticity of demand with $\alpha = 1$ dan $\alpha = \frac{1}{2}$ at the price p = 2.5.

Following is the completion of **Example 4.2** for $\alpha = 1$ and $\alpha = \frac{1}{2}$.

(a) For α = 1 and p = 2.5, we obtained:

$Q(p) = 180 - 9p^2 \rightarrow \frac{dQ}{dp} = -18p$

$E(p) = \left(\frac{p}{180 - 9p^2}\right) \cdot (-18p)$

$E(2.5) = \left(\frac{2.5}{180 - 9\cdot(2,5)^2}\right) \cdot (-18 \cdot 2.5) = -0.9090909090.$

We have $|E(2.5)| = 0.9090909090 < 1$, the price elasticity of demand in **Example 4.2** to be called Elastic for $\alpha = 1$ and $p = 2.5$. So, if the price rises (decreases) by 1%, the number of units of goods demanded will decrease (increase) as less than 1% (the percentage change in the number of units of goods requested is smaller than the percentage change in price per unit, meaning that the demand for the number of units of goods is not sensitive to changes in price). The percentage change in the number of units of goods requested is greater than the percentage change in price per unit, meaning that the demand for the number of units of goods is very sensitive to changes in price per unit.

(b) For $\alpha = \frac{1}{2}$ and p = 2.5, we obtained:

$Q(p) = 180 - 9p^2$

$D^{\alpha}Q(p) = D^{\frac{1}{2}}\,180 - D^{\frac{1}{2}}\,9p^2,$

$D^{\frac{1}{2}}Q(p) = 0 - 9\,\frac{\Gamma(2+1)}{\Gamma(2-\frac{1}{2}+1)}\,p^{2-\frac{1}{2}} = -9\,\frac{\Gamma(3)}{\Gamma(\frac{5}{2})}\,p^{\frac{3}{2}} = -9\,\frac{8}{3\sqrt{\pi}}\,p^{\frac{3}{2}} = -\frac{24}{\sqrt{\pi}}\,p^{\frac{3}{2}}$

$E_{\frac{1}{2}}(Q(p), p) = \frac{p}{Q(p)}\,\frac{d^{\frac{1}{2}}Q(p)}{d^{\frac{1}{2}}p} = \left(\frac{p}{180 - 9p^2}\right)\left(-\frac{24}{\sqrt{\pi}}\,p^{\frac{3}{2}}\right)$

$E_{\frac{1}{2}}(2.5) = \left(\frac{2.5}{180 - 9\cdot(2,5)^2}\right)\left(-\frac{24}{1.77245}\,(2.5)^{\frac{3}{2}}\right) = (0.02020202)\left(-\frac{24}{1.77245}\,(2.5)^{\frac{3}{2}}\right) = -1.081287343.$

$E_{\frac{1}{2}}(2.5)$ We have $|E_{\frac{1}{2}}(2.5)| = 1.081287343 > 1$ so that the price elasticity of demand in **Example 4.2** is called elastic for $\alpha = \frac{1}{2}$ and $p = 2.5$. So, if the price increase (decreases) by 1%, the amount the units of goods requested will decrease (increase) greater than 1 %. The percentage change in the number of units of goods requested is smaller than the percentage change in price per unit, meaning that demand is very sensitive to changes in price per unit.

4.5 Results of Numerical Simulation

Numerical results of $|E(p)|$ for the Example 4.2 by Theorem 4.2 using maple software for $p = 1,2,2.5,3,4$ and $\alpha = 1, \frac{99}{100}, \frac{75}{100}, \frac{50}{100}, \frac{10}{100}, \frac{1}{100}$, are given in Table 1. Whereas, Figure 1 represents values of $|Ep|$ for $0 \leq p \leq 4$. The number of goods demanded being positive or $Q(p) = 180 - 9p^2 > 0$, if $p > \sqrt{20}$.

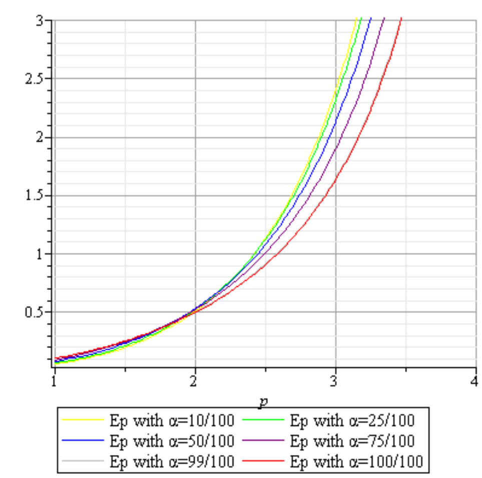

Fig. 1. Curve of $E(p)$ with $\alpha = 1, \frac{99}{100}, \frac{75}{100}, \frac{50}{100}, \frac{10}{100}, \frac{1}{100}$, for $1 \leq p \leq 4$, $k_0 = 180, k_1 = 0$ and $k_3 = -9$

Table 1
Numerical results of $|E(p)|$ for the various value of p and α

α	p	$\|Ep\|$	α	p	$\|Ep\|$
10/100	1	0.05760410	75/100	1	0.09290632
	2	0.51059205		2	0.52480311
	2.5	1.13482885		2.5	1.00893042
	3	2.40694409		3	1.90076767
	4	15.24477553		4	9.98559363
25/100	1	0.06544753	99/100	1	0.10481568
	2	0.52282866		2	0.50133751
	2.5	1.12377460		2.5	0.91355902
	3	2.31919706		3	1.64740707
	4	14.06863210		4	8.07719341
50/100	1	0.07918450	1	1	0.10526315
	2	0.53192304		2	0.50000000
	2.5	1.08128734		2.5	0.90909090
	3	2.13208368		3	1.63636363
	4	12.03604445		4	8.00000000

5. Conclusion

In conclusion, the numerical simulation results in Example 2, the calculation of the fractional price elasticity of demand with α = 1 and $\alpha = \frac{1}{2}$ showed significantly different results for $p = 2.5$. The sensitivity level of changes in the number of goods demanded by consumers towards price changes per unit for α =1 is not sensitive. Whereas the sensitivity level of changes in the number of goods demanded by consumers towards price changes per unit for $\alpha = \frac{1}{2}$ is very sensitive. In other words, the price elasticity of demand does not only depend on the latest price (current price) but changes in all prices from a specific time interval (with memory effects). The findings of this study suggest future studies can examine the phenomenon of market equilibrium using fractional-order derivatives.

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